

LIFTINGS OF NICHOLS ALGEBRAS OF DIAGONAL TYPE

II. ALL LIFTINGS ARE COCYCLE DEFORMATIONS

IVÁN ANGIOÑO; AGUSTÍN GARCÍA IGLESIAS

ABSTRACT. Let V be a braided vector space of diagonal type with a principal realization in the category of Yetter-Drinfeld modules of a cosemisimple Hopf algebra H and such that the Nichols algebra $\mathfrak{B}(V)$ is finitely presented. We show that every lifting of V is a cocycle deformation of $\mathfrak{B}(V)\#H$. In particular, it follows that every finite-dimensional pointed Hopf algebra A with abelian group of group-like elements Γ is a cocycle deformation of $\mathfrak{B}(V)\#\mathbb{k}\Gamma$, where $V \in {}^{\mathbb{k}\Gamma}_{\mathbb{k}\Gamma}\mathcal{YD}$ denotes the infinitesimal braiding of A .

In addition, we explain how to reduce the lifting problem to the connected case.

1. INTRODUCTION

This article belongs to the series started in [AAG], with the aim of computing all liftings of finite-dimensional Nichols algebras of diagonal type over an algebraically closed field of characteristic zero \mathbb{k} . We refer the reader to loc.cit. for a detailed description of this program. We fix

- A cosemisimple Hopf algebra H .
- A braided vector space of diagonal type (V, c) of dimension $\theta \in \mathbb{N}$ with a principal realization $(g_i, \chi_i)_{i \in \mathbb{I}_\theta}$ in ${}^H_H\mathcal{YD}$, such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional.

We recall that a Hopf algebra L is a *lifting* of $V \in {}^H_H\mathcal{YD}$ if $\text{gr } L \simeq \mathfrak{B}(V)\#H$. Our main result is the following:

Theorem 1.1. *Let H be a cosemisimple Hopf algebra and let (V, c) be a braided vector space of diagonal type such that $\dim \mathfrak{B}(V) < \infty$. Let L be a Hopf algebra such that its infinitesimal braiding is given by a principal realization $V \in {}^H_H\mathcal{YD}$. Then L is a cocycle deformation of $\mathfrak{B}(V)\#H$.*

Proof. Every such L is a lifting of V by [A1, Theorem 2]. The result follows by [AAG, Theorem 3.5], using Proposition 3.8. \square

Now, Theorem 1.1, combined with the results in [AAG, §3], reduces the lifting problem to an algorithm, cf. [AAG, §3.3]. Actually, this applies to a more general context: we assume that the ideal $\mathcal{J}(V)$ defining the Nichols

2010 *Mathematics Subject Classification.* 16T05.

The work was partially supported by CONICET, FONCyT-ANPCyT, Secyt (UNC), the MathAmSud project GR2HOPF.

algebra $\mathfrak{B}(V)$ is generated by a finite set \mathcal{G} —that is, we do not assume that $\dim \mathfrak{B}(V) < \infty$. To describe this algorithm, we fix

$$(1.1) \quad \Gamma = \langle g_1, \dots, g_\theta \rangle < Z(G(H)).$$

and let $\{x_1, \dots, x_\theta\}$ be a basis of V so that $x_i \in V_{g_i}^{\chi_i} = V_{g_i} \cap V^{\chi_i}$, $i \in \mathbb{I}_\theta$, cf. (2.1). We may assume that \mathcal{G} is composed of \mathbb{Z}^θ -homogeneous elements, hence for each relation $r \in \mathcal{G}$ of degree (a_1, \dots, a_θ) , then $r \in T(V)_{g_r}^{\chi_r}$ where

$$(1.2) \quad g_r := g_1^{a_1} \dots g_\theta^{a_\theta} \in \Gamma, \quad \chi_r := \chi_1^{a_1} \dots \chi_\theta^{a_\theta} \in \text{Alg}(H, \mathbb{k}).$$

The algorithm produces, for each family of scalars λ in the set

$$(1.3) \quad \mathcal{R} := \{\lambda = (\lambda_r)_{r \in \mathcal{G}} \in \mathbb{k}^\mathcal{G} : \lambda_r = 0 \text{ if } \chi_r \neq \epsilon \text{ or } g_r = 1\},$$

a cocycle deformation $\mathfrak{u}(\lambda)$ of $\mathfrak{B}(V) \# H$ such that

$$\text{gr } \mathfrak{u}(\lambda) \simeq \mathfrak{B}(V) \# H.$$

See §4.1 for a precise definition of the algebras $\mathfrak{u}(\lambda)$.

Theorem 1.2. *Let H be a cosemisimple Hopf algebra and let (V, c) be a braided vector space of diagonal type such that the ideal $\mathcal{J}(V)$ is generated by a finite set \mathcal{G} . If L is lifting of V , then there is $\lambda \in \mathcal{R}$ such that $L \simeq \mathfrak{u}(\lambda)$.*

Proof. By Proposition 3.8, [AAG, Theorem 3.5] holds for any such (V, c) . \square

See Theorem 5.3 for the description of the isoclasses of the algebras $\mathfrak{u}(\lambda)$. Observe that when $H = \mathbb{k}\Gamma$, Theorem 1.1 is equivalent to the following.

Corollary 1.3. *Let A be a finite-dimensional pointed Hopf algebra with abelian coradical. Then A is a cocycle deformation of $\text{gr } A$ and there is $\lambda \in \mathcal{R}$ such that $A \simeq \mathfrak{u}(\lambda)$.* \square

Finally, in §4 and §5 we contribute to the results in [AAG] by relating the case of multiple diagrams with the connected case, both for the definition of the liftings and the isomorphism classes, respectively.

ACKNOWLEDGMENTS

We thank Nicolás Andruskiewitsch for his constant support and council. We also thank Cristian Vay for pointing to us a mistake in a previous version of this article.

2. PRELIMINARIES

We work over an algebraically closed field of characteristic zero \mathbb{k} . If A is a \mathbb{k} -algebra and $S \subseteq A$ is a set, then we denote by $\langle S \rangle$ the ideal generated by S in A . Idem for a group G and a subset $S \subseteq G$, mutatis mutandis. We denote by $\text{Alg}(A, \mathbb{k})$ the set of algebra maps $A \rightarrow \mathbb{k}$.

If Γ is an abelian group, then we denote by $\widehat{\Gamma}$ the group of characters of Γ . If $n \in \mathbb{N}$, then we set $\mathbb{I}_n := \{1, \dots, n\}$, $\mathbb{I}_n^\circ := \{0, 1, \dots, n\}$.

If H is a Hopf algebra, then we denote by $(H_n)_{n \geq 0}$ the coradical filtration of H and set $\text{gr } H = \bigoplus_{n \geq 0} H_n / H_{n-1}$ the associated graded coalgebra; here

$H_{-1} := 0$. We denote by $\mathcal{P}(H)$ the subspace of primitive elements in H . If H' is another Hopf algebra, then $\text{Isom}(H, H')$ denotes the set of Hopf algebra isomorphisms $H \rightarrow H'$. We write ${}_H\mathcal{M}$ for the category of H -modules; an H -module algebra is thus an algebra in ${}_H\mathcal{M}$. In turn, ${}_H^H\mathcal{YD}$ stands for the category of Yetter-Drinfeld modules over H . Given $V \in {}_H^H\mathcal{YD}$, $g \in G(H)$ and $\chi \in \text{Alg}(H, \mathbb{k})$, then we set

$$(2.1) \quad V_g := \{x \in V \mid \delta(x) = g \otimes x\}, \quad V^\chi := \{x \in V \mid h \cdot x = \chi(h)x, h \in H\}.$$

If $V \in {}_H^H\mathcal{YD}$, then we denote by $\mathfrak{B}(V)$ the *Nichols algebra* of V . A *lifting* of $V \in {}_H^H\mathcal{YD}$ is a Hopf algebra L such that $\text{gr } L \simeq \mathfrak{B}(V) \# H$. A *lifting map* is a Hopf algebra projection $\phi : T(V) \# H \twoheadrightarrow L$ such that

$$(2.2) \quad \phi|_H = \text{id}_H \quad \text{and} \quad \phi|_{V \# H} : V \# H \xrightarrow{\simeq} L_1 \quad \text{as Hopf bimodules over } H.$$

2.1. Diagonal braidings. Let (V, c) be a braided vector space. We set $\theta = \dim V$, $\mathbb{I} = \mathbb{I}_\theta$. Assume that V is of diagonal type, with braiding matrix $q = (q_{ij})_{i,j \in \mathbb{I}}$; i.e. there is a basis $\{x_1, \dots, x_\theta\}$ such that $c = c^q$ satisfies:

$$c^q(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad i, j \in \mathbb{I}.$$

Let \mathcal{D} be the generalized Dynkin diagram associated to V as in [H] and let $\mathcal{D} = \bigsqcup_{i=1}^m \mathcal{D}_i$ be the decomposition of \mathcal{D} into m connected components, $m \in \mathbb{N}$. Given two vertices $i, j \in \mathbb{I}$ of \mathcal{D} , then we write $i \sim j$ if they belong to the same connected component and $i \not\sim j$ otherwise. If $V = \bigoplus_{i=1}^m V_i$ is the corresponding decomposition of V into sub-braided vector spaces V_i , associated to the connected diagram \mathcal{D}_i , $i \in \mathbb{I}_m$, then we refer to each V_i , $i \in \mathbb{I}_m$, as a (*connected*) *component* of V . We shall denote by

$$(2.3) \quad \mathcal{G}(0) := \{x_i x_j - q_{ij} x_j x_i \mid i < j \text{ and } i \not\sim j\}$$

the set of q -commutators of vertices in different components.

We denote by \mathcal{G} a set of generators of the ideal defining the Nichols algebra \mathfrak{B}_q . If every connected component of V belongs to the list of [H], then we take \mathcal{G} as computed in [A1] for each component, union $\mathcal{G}(0)$.

Let H be a Hopf algebra. A principal YD-realization of V over H is a family $(\chi_i, g_i)_{i \in \mathbb{I}} \in \text{Alg}(H, \mathbb{k}) \times G(H)$ such that $\chi_i(g_j) = q_{ji}$, $i, j \in \mathbb{I}$, and $\chi_i(h)g_i = \chi_i(h_{(2)})h_{(1)}g_i\mathcal{S}(h_{(3)})$ for all $h \in H$, $i \in \mathbb{I}$. In particular, it follows that $\Gamma = \langle g_1, \dots, g_\theta \rangle < Z(G(H))$, see (1.1). These data realize (V, c^q) as an object in ${}_H^H\mathcal{YD}$ in such a way that c^q coincides with the categorical braiding of ${}_H^H\mathcal{YD}$ and $x_i \in V_{g_i}^{\chi_i} = V_{g_i} \cap V^{\chi_i}$, $i \in \mathbb{I}$, cf. (2.1).

2.2. The strategy. We recall the setting of a generic step in the algorithm behind our strategy to compute the liftings as cocycle deformations. We refer the reader to [A+, AAG] for details. We fix

- A cosemisimple Hopf algebra H .
- $V \in {}_H^H\mathcal{YD}$ such that the ideal $\mathcal{J}(V)$ defining the Nichols algebra $\mathfrak{B}(V)$ is generated by a finite set \mathcal{G} .

Let $\mathcal{G} = \mathcal{G}_0 \sqcup \dots \sqcup \mathcal{G}_\ell$ be an *stratification* of \mathcal{G} cf. [A+, §5.1]; in particular \mathcal{G}_k is a basis of a Yetter-Drinfeld submodule $\mathcal{M}_k \subset \mathcal{P}(\mathfrak{B}_k)$, for each¹ k . We set $\mathfrak{B}_0 = T(V)$ and denote by $\mathfrak{B}_k = \mathfrak{B}_{k-1}/\langle \mathcal{G}_{k-1} \rangle$, $k \in \mathbb{I}_{\ell+1}$ the corresponding pre-Nichols algebras.

Set $\mathcal{T} = T(V) \# H$, $\mathcal{H}_k = \mathfrak{B}_k \# H$, $0 \leq k \leq \ell + 1$. Fix $k > 0$ and let $\mathcal{A}_{k-1} \in \text{Cleft}(\mathcal{H}_{k-1})$ be a cleft object resulting from the algorithm; we start with $\mathcal{A}_0 = \mathcal{T}$. Our aim is to find all $\mathcal{A}_k \in \text{Cleft}'(\mathcal{H}_k)$ as quotients of \mathcal{A}_{k-1} . This is depicted in the following *snapshot*² from [A+, p. 696]:

$$(2.4) \quad \begin{array}{ccccc} \mathcal{T} & \xrightarrow{\gamma_0 = \text{id}} & \mathcal{T} & \rightsquigarrow & L(\mathcal{T}, \mathcal{T}) \simeq \mathcal{T} \\ \downarrow \pi_{k-1} & & \downarrow \tau_{k-1} & & \\ \mathcal{H}_{k-1} & \xrightarrow{\gamma_{k-1}} & \mathcal{A}_{k-1} & \rightsquigarrow & L(\mathcal{A}_{k-1}, \mathcal{H}_{k-1}) \\ \downarrow \pi'_k & & \downarrow \tau'_k & & \\ \mathcal{H}_k & \xrightarrow{\gamma_k} & \mathcal{A}_k & \rightsquigarrow & L(\mathcal{A}_k, \mathcal{H}_k) \end{array}$$

(Note: In the original image, there are curved arrows on the left and right sides of the diagram. On the left, a curved arrow from \mathcal{T} to \mathcal{H}_k is labeled $\pi_k \equiv$. On the right, a curved arrow from \mathcal{T} to \mathcal{A}_k is labeled $\tau_k \equiv$.)

The maps γ_{k-1} , γ_k denote the corresponding sections, and the vertical arrows represent the natural (Hopf) algebra projections. In turn, the right-most column describes the Schauenburg left Hopf algebra $L(-, -)$ associated to each pair $(\mathcal{H}_k, \mathcal{A}_k)$, see [S].

Each cleft object \mathcal{A}_{k-1} can be decomposed as $\mathcal{A}_{k-1} = \mathcal{E}_{k-1} \# H$, where $\mathcal{E}_0 = T(V)$ and each \mathcal{E}_{k-1} is an H -module algebra [A+, Proposition 5.8].

As a \mathbb{k} -linear space \mathcal{E}_k is defined as the quotient

$$(2.5) \quad \mathcal{E}_k = \mathcal{E}_k(\boldsymbol{\lambda}) = \mathcal{E}_{k-1} / \langle \gamma_{k-1}(r) - \lambda_r : r \in \mathcal{G}_{k-1} \rangle,$$

for each family $\boldsymbol{\lambda} = (\lambda_r)_{r \in \mathcal{G}} \in \mathbb{k}$. If $\mathcal{E}_k(\boldsymbol{\lambda}) \neq 0$, then it is a \mathbb{k} -algebra. If, additionally, \mathcal{E}_k is an H -module algebra, then $\mathcal{A}_k(\boldsymbol{\lambda}) := \mathcal{E}_k \# H \in \text{Cleft}(\mathcal{H}_k)$, by [A+, Proposition 5.8].

Next lemma indicates when \mathcal{E}_k is an H -module algebra. In §3 we study when $\mathcal{E}_k(\boldsymbol{\lambda}) \neq 0$ for the case of diagonal braidings.

We introduce some terminology first. For each $r \in \mathcal{G}_{k-1}$ and each $h \in H$, we write the H -action as $h \cdot r = \sum_{s \in \mathcal{G}_{k-1}} \mu_{rs}(h)s$, some $(\mu_{rs}(h))_{s \in \mathcal{G}_{k-1}} \in \mathbb{k}$.

Lemma 2.1. *Assume $\mathcal{E}_k \neq 0$; then \mathcal{E}_k is an H -module algebra if and only if*

$$(2.6) \quad \epsilon(h)\lambda_r = \sum_{s \in \mathcal{G}_{k-1}} \mu_{rs}(h)\lambda_s, \quad \forall r \in \mathcal{G}_{k-1}, h \in H.$$

Proof. If (2.6) holds, then the ideal $\langle \gamma_{k-1}(r) - \lambda_r : r \in \mathcal{G}_{k-1} \rangle \subset \mathcal{E}_{k-1}$ is an H -submodule of \mathcal{E}_{k-1} and hence the lemma follows. \square

¹ \mathcal{G}_ℓ can be chosen so that it is not properly contained in $\mathcal{P}(\mathfrak{B}_\ell)$, see loc.cit.

²The term is due to Cristian Vay.

3. ALL LIFTINGS ARE COCYCLE DEFORMATIONS

In this section we further assume that

- $V \in {}^H_H\mathcal{VD}$ comes from a principal realization $(g_i, \chi_i)_{i \in \mathbb{I}_\theta}$ of a braided vector space of diagonal type (V, c) with matrix \mathbf{q} .

We show that every lifting of V is a cocycle deformation of $\mathfrak{B}_{\mathbf{q}} \# H$, by showing that the algebras $\mathcal{E}_k \in {}_H\mathcal{M}$ in (2.5) are nonzero.

We consider \mathcal{G} as in §2.1. We set $\Gamma < Z(G(H))$ as in (1.1), so each $r \in \mathcal{G}$ belongs to a unique component $r \in T(V)_{g_r}^{\chi_r}$, for some $g_r \in \Gamma$ and $\chi_r \in \text{Alg}(H, \mathbb{k})$, cf. (1.2). We fix an stratification $\mathcal{G}_0 \sqcup \cdots \sqcup \mathcal{G}_\ell$ in such a way that $\mathcal{G}_0 = \mathcal{G}(0)$ as in (2.3) and such that $\mathbb{k}\langle \mathcal{G}_\ell \rangle$ is the normal braided Hopf subalgebra of \mathfrak{B}_ℓ generated by the powers of the Cartan root vectors, cf. [A2, Theorem 31].

In this context, Lemma 2.1 is equivalent to the following.

Corollary 3.1. *If $\mathcal{E}_k \neq 0$, then \mathcal{E}_k is an H -module algebra if and only if*

$$(3.1) \quad \lambda_r = 0 \quad \text{if } \chi_r \neq \epsilon, \quad \text{for all } r \in \mathcal{G}_{k-1}.$$

Proof. In this setting, (2.6) becomes (3.1). \square

Next we show that condition (3.1) also guarantees that $\mathcal{E}_k(\boldsymbol{\lambda}) \neq 0$. We begin with a technical lemma.

Lemma 3.2. *Let $\boldsymbol{\lambda} = (\lambda_r)_{r \in \mathcal{G}} \in \mathbb{k}$ be a family of scalars satisfying (3.1). For each $k \geq 0$, $\mathcal{E} = \mathcal{E}_k(\boldsymbol{\lambda})$ decomposes as a sum of Γ -eigenspaces $\mathcal{E} = \bigoplus_{\chi \in \widehat{\Gamma}} \mathcal{E}^\chi$.*

Proof. Since $V = \bigoplus_{i \in \mathbb{I}_\theta} V^{\chi_i}$, then the H -module algebra $\mathcal{E}_0 = T(V)$ decomposes as a sum of Γ -eigenspaces $T(V) = \bigoplus_{\chi \in \widehat{\Gamma}} T(V)^\chi$. Fix $k > 0$ and assume that the statement holds for \mathcal{E}_{k-1} , we show that it also holds for \mathcal{E}_k . By hypothesis, the ideal defining \mathcal{E}_k , see (2.5), is generated by $\widehat{\Gamma}$ -homogeneous elements. Hence the lemma follows. \square

The following result will be useful.

Lemma 3.3. *Let $k \leq \ell$, $\boldsymbol{\lambda} \in \mathcal{R}$. Assume that the following conditions hold:*

$$\mathcal{E}_{k-1} / \langle \gamma_{k-1}(r) : r \in \mathcal{G}_{k-1} \rangle \neq 0, \quad \mathfrak{B}_{k-1} / \langle r - \lambda_r : r \in \mathcal{G}_{k-1} \rangle \neq 0.$$

Then $\mathcal{E}_k(\boldsymbol{\lambda}) \neq 0$.

Proof. We consider the algebra map $\varrho : \mathcal{E}_{k-1} \rightarrow \mathcal{E}_{k-1} \otimes \mathfrak{B}_{k-1}$ given by the braided coaction as in [AG, Lemma 4.1]. As $\mathcal{G}_{k-1} \subset \mathcal{P}(\mathfrak{B}_{k-1})$,

$$\varrho(\gamma_{k-1}(r)) = \gamma_{k-1}(r) \otimes 1 + 1 \otimes r, \quad r \in \mathcal{G}_{k-1}.$$

Now ϱ induces an algebra map

$$\varrho' : \mathcal{E}_{k-1} \rightarrow \mathcal{E}_{k-1} / \langle \gamma_{k-1}(r) : r \in \mathcal{G}_{k-1} \rangle \otimes \mathfrak{B}_{k-1} / \langle r - \lambda_r : r \in \mathcal{G}_{k-1} \rangle$$

such that $\varrho'(\gamma_{k-1}(r) - \lambda_r) = 0$, $r \in \mathcal{G}_{k-1}$. Hence $\mathcal{E}_k(\boldsymbol{\lambda}) \neq 0$. \square

Lemma 3.4. *If $\widetilde{\mathcal{E}}_{\mathbf{q}}(\boldsymbol{\lambda}) := \mathcal{E}_\ell(\boldsymbol{\lambda}) \neq 0$, then $\mathcal{E}_{\ell+1}(\boldsymbol{\lambda}) \neq 0$.*

Proof. We recall that the normal Hopf subalgebra $\mathbb{k}\langle\mathcal{G}_\ell\rangle \subset \mathfrak{B}_\ell$ is a q -polynomial algebra [A2, Proposition 21]. Hence, by [A+, Theorem 3.1], also [Gu, Theorem 4], $\mathcal{E}_{\ell+1}(\boldsymbol{\lambda}) \neq 0$. \square

We fix $\mathcal{D} = \bigsqcup_{i=1}^m \mathcal{D}_i$ and respectively $V = \bigoplus_{i=1}^m V_i$ the decompositions of \mathcal{D} and V into connected components. We denote by $\mathcal{G}(i)$ the set of generators of the ideal defining the Nichols algebra $\mathfrak{B}(V_i)$, $i \in \mathbb{I}_m$. As a result, the set \mathcal{G} of generators of $\mathcal{J}(V)$, see §2.1, decomposes as a disjoint union, cf. (2.3):

$$(3.2) \quad \mathcal{G} = \mathcal{G}(0) \sqcup \mathcal{G}(1) \sqcup \cdots \sqcup \mathcal{G}(m).$$

Let $\mathcal{R} \subseteq \mathbb{k}^{\mathcal{G}}$, resp. $\mathcal{R}(i) \subseteq \mathbb{k}^{\mathcal{G}(i)}$, $i \in \mathbb{I}_m$, be the set of deformation parameters for V , resp. V_i , cf. (1.3). Then we have a restriction map:

$$(3.3) \quad \mathcal{R} \rightarrow \mathcal{R}(i), \quad \boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}_{|i} := (\lambda_r)_{r \in \mathcal{G}(i)} \in \mathcal{R}(i).$$

In particular, $\boldsymbol{\lambda}$ splits into a product of subfamilies

$$(3.4) \quad \boldsymbol{\lambda} = \boldsymbol{\lambda}_{|0} \times \boldsymbol{\lambda}_{|1} \times \cdots \times \boldsymbol{\lambda}_{|m}$$

in such a way that

- (i) $\boldsymbol{\lambda}_{|0} \in \mathbb{k}^{\mathcal{G}(0)}$ is the subset of deformation parameters linking vertices in different connected components;
- (ii) $\boldsymbol{\lambda}_{|i} \in \mathbb{k}^{\mathcal{G}(i)}$ is the subset of deformation parameters for the i th connected component of V , $i \in \mathbb{I}_m$.

Lemma 3.5. *Let $\boldsymbol{\lambda} = (\lambda_r)_{r \in \mathcal{G}} \in \mathbb{k}$ be a family of scalars satisfying (3.1). If $\mathcal{E}(\boldsymbol{\lambda}_{|i}) \neq 0$ for every $i \in \mathbb{I}_m$ and $\boldsymbol{\lambda}_{|0} = \mathbf{0}$, then $\tilde{\mathcal{E}}_q(\boldsymbol{\lambda}) \neq 0$.*

Proof. Let $\rho : T(V) \rightarrow \text{End} \otimes_{t=1}^m \mathcal{E}(\boldsymbol{\lambda}_{|t})$ be the algebra map such that

$$\begin{aligned} \rho(y_k)(\mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_m) &:= g_k \cdot \mathbf{y}_1 \otimes \cdots \otimes g_k \cdot \mathbf{y}_{i-1} \otimes y_k \mathbf{y}_i \otimes \mathbf{y}_{i+1} \otimes \cdots \otimes \mathbf{y}_m, \\ &\quad \mathbf{y}_t \in \mathcal{E}(\boldsymbol{\lambda}_{|t}), \quad k \in \mathcal{D}_i, \quad i \in \mathbb{I}_m. \end{aligned}$$

We claim that $\rho(y_{k\ell}) = 0$ for all $k \not\sim \ell$, $k < \ell$. Indeed, if $k \in \mathcal{D}_i$, $\ell \in \mathcal{D}_j$, $i < j \in \mathbb{I}_m$, then

$$\begin{aligned} \rho(y_k y_\ell)(\mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_m) &:= g_k g_\ell \cdot \mathbf{y}_1 \otimes \cdots \otimes g_k g_\ell \cdot \mathbf{y}_{i-1} \otimes y_k(g_\ell \cdot \mathbf{y}_i) \\ &\quad \otimes g_\ell \cdot \mathbf{y}_{i+1} \otimes \cdots \otimes g_\ell \cdot \mathbf{y}_{j-1} \otimes y_\ell \mathbf{y}_j \otimes \mathbf{y}_{j+1} \otimes \cdots \otimes \mathbf{y}_m, \\ \rho(y_\ell y_k)(\mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_m) &:= g_k g_\ell \cdot \mathbf{y}_1 \otimes \cdots \otimes g_k g_\ell \cdot \mathbf{y}_{i-1} \otimes g_\ell \cdot (y_k \mathbf{y}_i) \\ &\quad \otimes g_\ell \cdot \mathbf{y}_{i+1} \otimes \cdots \otimes g_\ell \cdot \mathbf{y}_{j-1} \otimes y_\ell \mathbf{y}_j \otimes \mathbf{y}_{j+1} \otimes \cdots \otimes \mathbf{y}_m. \end{aligned}$$

As $k \not\sim \ell$, $q_{\ell k} q_{k\ell} = 1$ and hence

$$g_\ell \cdot (y_k \mathbf{y}_i) = (g_\ell \cdot y_k)(g_\ell \cdot \mathbf{y}_i) = q_{\ell k} y_k(g_\ell \cdot \mathbf{y}_i) = q_{k\ell}^{-1} y_k(g_\ell \cdot \mathbf{y}_i),$$

so $\rho(y_{k\ell})(\mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_m) = 0$ for all $\mathbf{y}_t \in \mathcal{E}(\boldsymbol{\lambda}_{|t})$.

Let $r \in \mathcal{G}(p)$. We claim that $\rho(\gamma_{p-1}(r))(\mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_m) = 0$. It suffices to consider $\mathbf{y}_t \in \mathcal{E}(\boldsymbol{\lambda}_{|t})$ such that $\mathbf{y}_t \in \pi_t(T(V_t)_{\mathbf{g}_t}^{\chi_t})$ for some $\mathbf{g}_t \in \Gamma$, $\chi_t \in \widehat{\Gamma}$, where $\pi_t : T(V_t) \rightarrow \mathcal{E}(\boldsymbol{\lambda}_{|t})$ is the canonical projection. For each $k \in \mathcal{D}_i$,

$$\rho(y_k)(\mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_m) = \chi_1 \cdots \chi_{i-1}(g_k) \mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_{i-1} \otimes y_k \mathbf{y}_i \otimes \mathbf{y}_{i+1} \otimes \cdots \otimes \mathbf{y}_m$$

$$= \chi_k^{-1}(\mathbf{g}_1 \dots \mathbf{g}_{i-1}) \mathbf{y}_1 \otimes \dots \otimes \mathbf{y}_{i-1} \otimes y_k \mathbf{y}_i \otimes \mathbf{y}_{i+1} \otimes \dots \otimes \mathbf{y}_m.$$

Since $\gamma_{p-1}(r) \in \mathcal{E}(\boldsymbol{\lambda}_{|p})^{\chi_r}$, we have

$$\begin{aligned} \rho(\gamma_{p-1}(r))(\mathbf{y}_1 \otimes \dots \otimes \mathbf{y}_m) &= \\ \chi_r^{-1}(\mathbf{g}_1 \dots \mathbf{g}_{i-1}) \mathbf{y}_1 \otimes \dots \otimes \mathbf{y}_{i-1} \otimes \gamma_{p-1}(r) \mathbf{y}_i \otimes \mathbf{y}_{i+1} \otimes \dots \otimes \mathbf{y}_m &= 0. \end{aligned}$$

Hence ρ induces an algebra map $\tilde{\rho} : \tilde{\mathcal{E}}_q(\boldsymbol{\lambda}) \rightarrow \text{End} \otimes_{t=1}^m \mathcal{E}(\boldsymbol{\lambda}_{|t})$ and therefore $\tilde{\mathcal{E}}_q(\boldsymbol{\lambda}) \neq 0$. \square

Lemma 3.6. *If $\boldsymbol{\lambda}_{|i} = \mathbf{0}$ for all $i > 0$, then $\tilde{\mathcal{E}}_q(\boldsymbol{\lambda}) \neq 0$.*

Proof. Set $N_j = \text{ord } \chi_j(g_j)$, $y_{jk} := y_j y_k - q_{jk} y_k y_j$, $j < k \in \mathbb{I}$. Let $\mathcal{F}(\boldsymbol{\lambda}_{|0})$ be the quotient of $T(V)$ by the ideal generated by

$$y_{jk} - \lambda_{jk}, \quad j \not\sim k, \quad y_{jk}, \quad j \sim k, j < k, \quad y_j^{N_j}, \quad j \in \mathbb{I}.$$

An easy application of the Diamond Lemma shows that $\mathcal{F}(\boldsymbol{\lambda}_{|0}) \neq 0$.

For each connected component V_i , the relations in $\mathcal{G}(i)$ hold in $\tilde{\mathcal{E}}_q$, since $\boldsymbol{\lambda}_{|i} = \mathbf{0}$, cf. (2.5). Recall that the set $\mathcal{G}(i)$ consists of relations $y_j^{N_j}$, for certain $j \in \mathbb{I}_{\theta_i}$, and relations involving at least two letters y_j, y_k contained in the ideal $\langle y_{jk} | j < k \in \mathbb{I}_{\theta_i} \rangle$. Hence $\tilde{\mathcal{E}}_q(\boldsymbol{\lambda})$ projects onto $\mathcal{F}(\boldsymbol{\lambda}_{|0})$. \square

Proposition 3.7. *Let $\boldsymbol{\lambda} = (\lambda_r)_{r \in \mathcal{G}} \in \mathbb{k}$ be a family of scalars satisfying (3.1). If $\mathcal{E}(\boldsymbol{\lambda}_{|i}) \neq 0$ for every $i \in \mathbb{I}_m$, then $\mathcal{E}(\boldsymbol{\lambda}) \neq 0$.*

Proof. The case $\boldsymbol{\lambda}_{|0} = \mathbf{0}$ follows by Lemma 3.5, while the case $\boldsymbol{\lambda}_{|i} = \mathbf{0}$ for all $i > 0$ is Lemma 3.6. Hence we make use of Lemma 3.3 to combine both cases and conclude that $\mathcal{E}(\boldsymbol{\lambda}) \neq 0$ for any $\boldsymbol{\lambda}$. \square

Proposition 3.8. *Let $\boldsymbol{\lambda} = (\lambda_r)_{r \in \mathcal{G}} \in \mathbb{k}$ be a family of scalars satisfying (3.1). Then $\mathcal{E}_k(\boldsymbol{\lambda}) \neq 0$, for every $k \geq 0$.*

Proof. We may restrict to the connected case by Proposition 3.7. Moreover, we just need to check that $\tilde{\mathcal{E}}_q(\boldsymbol{\lambda})$ as in Lemma 3.4 is nonzero. We compute this algebra following (2.5), by deforming the corresponding defining relations of \mathfrak{B}_q , as computed in [A1] for each diagram in the classification of [H]. This is done in the Appendix, by a case-by-case analysis of all Nichols algebras following the explicit list given in [AA]. \square

4. DIAGONAL CASE: NON-CONNECTED DIAGRAMS

We fix a pair (H, V) as in §2.1. We focus on the case in which the generalized Dynkin diagram \mathcal{D} associated to V is not connected.

We fix $\{x_1, \dots, x_\theta\}$ a basis of V with $x_i \in V_{g_i}^{\chi_i}$, $i \in \mathbb{I}$. In particular, every $r \in \mathcal{G}$ belongs to a component $T(V)_{g_r}^{\chi_r}$, for some $g_r \in \Gamma$, $\chi_r \in \text{Alg}(H, \mathbb{k})$.

4.1. The liftings $\mathbf{u}(\boldsymbol{\lambda})$. We fix $\boldsymbol{\lambda} \in \mathcal{R}$ as in (1.3). In [AAG, §3.1] we defined recursively a family of Hopf algebras $(\mathcal{L}_k(\boldsymbol{\lambda}))_{k \in \mathbb{I}_{\ell+1}^o}$ in such a way that $\mathcal{L}_k(\boldsymbol{\lambda}) \simeq L(\mathcal{A}_k(\boldsymbol{\lambda}), \mathcal{H}_k)$, cf. (2.4). We set

$$\mathbf{u}(\boldsymbol{\lambda}) := \mathcal{L}_{\ell+1}(\boldsymbol{\lambda}).$$

We set $\mathcal{L}_0 := T(V) \# H$. Fix $k > 0$ and assume that $\mathcal{L}_{k-1} = \mathcal{L}_{k-1}(\boldsymbol{\lambda})$ is already defined. In particular, there is a left coaction $\mathcal{A}_{k-1} \rightarrow \mathcal{L}_{k-1} \otimes \mathcal{A}_{k-1}$. Let $\gamma : \mathcal{H}_{k-1} \rightarrow \mathcal{A}_{k-1}$ be the section corresponding to the right coaction $\mathcal{A}_{k-1} \rightarrow \mathcal{A}_{k-1} \otimes \mathcal{H}_{k-1}$. Then

$$\nabla(r) := \gamma(r)_{(-1)} \otimes \gamma(r)_{(0)} - g_r \otimes \gamma(r) \in \mathcal{L}_{k-1} \otimes 1, \quad \text{for all } r \in \mathcal{G}_{k-1},$$

by [A+, Corollary 5.12] and thus $\nabla(r) = \tilde{r} \otimes 1$, some $\tilde{r} \in \mathcal{L}_{k-1}$. We set

$$(4.1) \quad \mathcal{L}_k = \mathcal{L}_k(\boldsymbol{\lambda}) := \mathcal{L}_{k-1} / \langle \tilde{r} - \lambda_r(1 - g_r) : r \in \mathcal{G}_{k-1} \rangle.$$

By [AAG, Proposition 3.3] combined with Proposition 3.8, $\mathcal{L}_k \simeq L(\mathcal{A}_k, \mathcal{H}_k)$.

By Theorem 1.2, $(\mathbf{u}(\boldsymbol{\lambda}))_{\boldsymbol{\lambda} \in \mathcal{R}}$ is the family of all liftings of V .

4.2. Non-connected diagrams. We fix $\mathcal{D} = \bigsqcup_{i=1}^m \mathcal{D}_i$ and respectively $V = \bigoplus_{i=1}^m V_i$ the decompositions of \mathcal{D} and V into connected components. We keep the notation as in §3. Notice that (3.3) associates a lifting $\mathbf{u}(\boldsymbol{\lambda}_{|i})$ of V_i to each lifting $\mathbf{u}(\boldsymbol{\lambda})$ of V . For each $i \in \mathbb{I}_m$, we set $\mathcal{J}(\boldsymbol{\lambda}_{|i}) \subset T(V_i)$ as the ideal such that

$$\mathbf{u}(\boldsymbol{\lambda}_{|i}) = T(V_i) \# H / \mathcal{J}(\boldsymbol{\lambda}_{|i}).$$

Let $\mathcal{G}(i) = \mathcal{G}(i)_0 \sqcup \cdots \sqcup \mathcal{G}(i)_{\ell(i)}$ be an stratification of the set of defining relations $\mathcal{G}(i)$ of $\mathfrak{B}(V_i)$. Then

$$(4.2) \quad \mathcal{G} = \mathcal{G}(1)_0 \sqcup \cdots \sqcup \mathcal{G}(1)_{\ell(1)} \sqcup \cdots \sqcup \mathcal{G}(m)_0 \sqcup \cdots \sqcup \mathcal{G}(m)_{\ell(m)} \sqcup \mathcal{G}(0)$$

is an stratification of \mathcal{G} with $N + 1$ steps, $N = r + \ell(1) + \cdots + \ell(m)$.

We denote by $\mathbf{u}_{|i}(\boldsymbol{\lambda})$ the subalgebra of $\mathbf{u}(\boldsymbol{\lambda})$ generated by $H \oplus V_i \# H$; this is indeed a Hopf subalgebra.

Lemma 4.1. $\mathbf{u}_{|i}(\boldsymbol{\lambda}) \simeq \mathbf{u}(\boldsymbol{\lambda}_{|i})$ as Hopf algebras.

Proof. Let $\phi : T(V) \# H \rightarrow \mathbf{u}(\boldsymbol{\lambda})$ be a lifting map, cf. §2. We assume that $i = 1$, for short. We consider the map $\phi_1 : T(V_1) \# H \rightarrow \mathbf{u}(\boldsymbol{\lambda})$ given by the composition of ϕ with the canonical inclusion $T(V_1) \# H \hookrightarrow T(V) \# H$. Using the stratification as in (4.2), it follows that $\phi_1(\mathcal{J}(\boldsymbol{\lambda}_{|1})) = 0$ and thus ϕ_1 factors through a map $\varphi : \mathbf{u}(\boldsymbol{\lambda}_{|1}) \rightarrow \mathbf{u}(\boldsymbol{\lambda})$.

By construction, the first term $\mathbf{u}(\boldsymbol{\lambda}_{|1})_1$ of the coradical filtration of $\mathbf{u}(\boldsymbol{\lambda}_{|1})$ is (isomorphic to) $H \oplus V_1 \# H$; similarly $\mathbf{u}(\boldsymbol{\lambda})_1 \simeq H \oplus V \# H$. Now, the restriction of φ to $\mathbf{u}(\boldsymbol{\lambda}_{|1})_1$ is the canonical inclusion $H \oplus V_1 \# H \hookrightarrow H \oplus V \# H$ since ϕ is a lifting map. Hence φ is injective by [Mo, Theorem 5.3.1]. \square

Now we proceed in the opposite direction, and show in Proposition 4.2 that we can combine the liftings of each of the V_i 's to find (all) liftings of V .

We denote by $\mathcal{J}(\lambda_{>0})$ the ideal of $T(V)\#H$ generated by $\bigcup_{i \in \mathbb{I}_m} \mathcal{J}(\lambda_{|i})$; here we consider the natural identification $T(V_i)\#H \hookrightarrow T(V)\#H$. We set

$$\mathfrak{U}(\lambda) = T(V)\#H / \mathcal{J}(\lambda_{>0}).$$

The subfamily $\lambda_{|0}$ consists of scalars $(\lambda_{i,j})_{i \not\sim j, i < j}$ subject to

$$\lambda_{i,j} = 0 \quad \text{if} \quad \chi_i \chi_j \neq \epsilon \text{ or } g_i g_j = 1,$$

cf. (3.1). We consider the Hopf ideal $\mathcal{J}(\lambda_{|0}) \subset \mathfrak{U}(\lambda)$ generated by

$$(4.3) \quad \text{ad}(x_i)(x_j) - \lambda_{i,j}(1 - g_i g_j), \quad i < j, i \not\sim j$$

and set $\mathfrak{u}'(\lambda) = \mathfrak{U}(\lambda) / \mathcal{J}(\lambda_{|0})$.

Proposition 4.2. *$\mathfrak{u}'(\lambda)$ is a lifting of V ; moreover, $\mathfrak{u}'(\lambda) \simeq \mathfrak{u}(\lambda)$.*

Proof. We fix an stratification as in (4.2). Then, for each step $k = 0, \dots, N$, there is a surjective Hopf algebra map $\mathcal{L}_k \twoheadrightarrow \mathfrak{U}(\lambda)$, cf. (4.1); hence there is a Hopf algebra projection $\tau' : \mathcal{L}_N \twoheadrightarrow \mathfrak{u}'(\lambda)$. Now, the final step $\mathcal{L}_{N+1} = \mathfrak{u}(\lambda)$ is given precisely by the quotient of \mathcal{L}_N by the relations (4.3). Therefore, τ' factors through a map $\tau : \mathfrak{u}(\lambda) \twoheadrightarrow \mathfrak{u}'(\lambda)$; we shall denote by τ_1 the restriction of τ to the first term $\mathfrak{u}(\lambda)_1 \simeq H \oplus V\#H$ of the coradical filtration of $\mathfrak{u}(\lambda)$.

Conversely, we fix a lifting map $\phi : T(V)\#H \twoheadrightarrow \mathfrak{u}(\lambda)$, cf. §2. Now, by Lemma 4.1, $\phi(\mathcal{J}(\lambda_{|i})) = 0$ for each $i \in \mathbb{I}_m$, and thus ϕ factors through a Hopf algebra epimorphism $\phi' : \mathfrak{U}(\lambda) \twoheadrightarrow \mathfrak{u}(\lambda)$. We observe that $\mathcal{G}(0)$ is composed of primitive elements in $T(V)$, hence relations (4.3) hold in $\mathfrak{u}(\lambda)$ by definition, see (4.1). Hence $\phi'(\mathcal{J}(\lambda_{|0})) = 0$ and this induces a Hopf algebra projection $\varphi : \mathfrak{u}'(\lambda) \twoheadrightarrow \mathfrak{u}(\lambda)$ such that $\varphi_1 := \varphi|_{H \oplus V\#H}$ is injective, by (2.2).

It follows that $\tau_1 \circ \varphi_1 = \varphi_1 \circ \tau_1 = \text{id}_{H \oplus V\#H}$, hence $\tau : \mathfrak{u}(\lambda) \rightarrow \mathfrak{u}'(\lambda)$ defines a Hopf algebra isomorphism, with inverse φ . \square

5. ON ISOMORPHISM CLASSES

Let (V, H) be as in §2.1. If $\lambda \in \mathcal{R}$, cf. (1.3), then we say that (H, V, λ) is a *lifting datum*. Let (V', H') a new pair as in §2.1; we denote by $((g'_i, \chi'_i))_{i \in \mathbb{I}_{\theta'}}$, $\theta' = \dim V'$, \mathcal{R}' , the corresponding input information.

When V, V' are connected, [AAG, Theorem 3.9] describes the set of isomorphisms classes $\text{Isom}(\mathfrak{u}(\lambda), \mathfrak{u}(\lambda'))$ in terms of certain symmetries of the lifting data. This is in turn inspired by [AS2, Theorem 7.2], where this is done for certain families of liftings of Cartan type with minor restrictions on the parameters; see also [AS1, Theorem 7.1] for some families of type A .

The claim of [AAG, Theorem 3.9] extends verbatim to the non-connected case, see Theorem 5.3; also Remark 5.2. In addition, we review this case in the spirit of Proposition 4.2, that is we study the isomorphism classes of the algebras $\mathfrak{u}(\lambda)$ in terms of the components of V , see Proposition 5.7.

Consider the decompositions $V = \bigoplus_{i=1}^m V_i$ and $V' = \bigoplus_{i=1}^{m'} V'_i$ of V and V' as sums of connected braided vector subspaces; set $\theta_i = \dim V_i$, $\theta'_j = \dim V_j$,

$i \in \mathbb{I}_m, j \in \mathbb{I}_{m'}$. We fix $\mathbb{I} = \mathbb{I}_\theta$ and set $\mathbb{I}(i) = \{j \in \mathbb{I} \mid g_j = g_i \text{ and } \chi_j = \chi_i\}$, for each $i \in \mathbb{I}$ and consider the groups

$$\begin{aligned} \mathbb{S}_q &= \{\sigma \in \mathbb{S}_\theta \mid q_{ij} = q_{\sigma(i)\sigma(j)} \forall i, j \in \mathbb{I}\}; \\ \mathbf{L} &= \{s \in \text{GL}_\theta(\mathbb{k}) \mid s_{ij} = 0 \text{ if } j \notin \mathbb{I}(i)\}. \end{aligned}$$

These groups act on \mathcal{R} [AAG, Lemmas 3.10 & 3.11]; we write each action as

$$(5.1) \quad \lambda \mapsto \lambda^\sigma, \quad \lambda \mapsto s \cdot \lambda; \quad \text{for } \lambda \in \mathcal{R}, \sigma \in \mathbb{S}_q, s \in \mathbf{L}.$$

Remark 5.1. Let $\lambda \in \mathcal{R}$. Then the actions (5.1) restrict to each component $\lambda|_i, i \geq 0$, as in (3.4), cf. [AAG, Lemmas 3.10 & 3.11], also [AAG, Example 3.12]. In particular,

$$(5.2) \quad (s \cdot \lambda^\sigma)|_0 = (s_{ij}^{-1} \lambda_{\sigma^{-1}(i), \sigma^{-1}(j)})_{i,j}, \quad \sigma \in \mathbb{S}_q, s \in \mathbf{L}.$$

Here, if $\lambda|_0 = (\lambda_r)_{r \in \mathcal{G}(0)}$, then we set $\lambda_{i,j} := \lambda_r$ for $i < j \in \mathbb{I}$ defining the q -commutator r cf. (2.3).

Remark 5.2. When V connected, the sets $\mathbb{I}(i)$ and \mathbf{L} are explicitly described in [AAG, §3.2.1]. This is indeed the only occurrence of the connectedness hypothesis in loc.cit.

We recall from [AAG, Lemma 3.7] that if $\psi \in \text{Isom}(\mathbf{u}(\lambda), \mathbf{u}(\lambda'))$, then $\varphi := \psi|_H \in \text{Isom}(H, H')$ and $T := \psi|_V: V \rightarrow V'$ is an isomorphism of braided vector spaces; so $\theta = \theta', m = m'$ and there is $s = (s_{ij}) \in \mathbf{L}$ with

$$T(a_i) = \sum_{j \in \mathbb{I}(\sigma(i))} s_{ij} a'_j, \quad i \in \mathbb{I},$$

where we denote by $\{a_i\}_{i \in \mathbb{I}}$, resp. $\{a'_i\}_{i \in \mathbb{I}}$, the images of the elements $x_i \in V$, resp. $x'_i \in V', i \in \mathbb{I}$, via the natural projections $T(V) \# H \rightarrow \mathbf{u}(\lambda)$, resp. $T(V') \# H' \rightarrow \mathbf{u}(\lambda')$. In turn, φ defines an element $\sigma \in \mathbb{S}_q$ by the identities

$$(5.3) \quad \varphi(g_j) = g'_{\sigma(j)} \quad \text{and} \quad \chi'_{\sigma(j)} \circ \varphi = \chi_j, \quad j \in \mathbb{I}.$$

These data actually determine the isomorphism ψ , see Theorem 5.3. We let $\text{Isom}(\lambda, \lambda')$ be the set of all triples $(\varphi, \sigma, s) \in \text{Isom}(H, H') \times \mathbb{S}_q \times \mathbf{L}$ satisfying (5.3) and such that $\lambda' = s \cdot \lambda^\sigma$.

Theorem 5.3. [AAG, Theorem 3.9] *Let (H, V, λ) and (H', V', λ') be two lifting data. Then $\text{Isom}(\mathbf{u}(\lambda), \mathbf{u}(\lambda')) \simeq \text{Isom}(\lambda, \lambda')$.* \square

Remark 5.4. If $s \in \mathbf{L}$ and $i \not\sim j \in \mathbb{I}$, then one of the following holds:

- (a) $s_{ij} = 0$;
- (b) $\mathbb{k}\{x_i\} \simeq \mathbb{k}\{x_j\} \simeq V_p$ for some $p \in \mathbb{I}_m$ and $q_{ii} = q_{ij} = q_{ji} = q_{jj} = -1$.

Indeed, given such i, j , if there is $k \neq i, j$ connected with one of them, say $k \sim i$, and $s_{ij} \neq 0$, then $j \in \mathbb{I}(i)$, that is $\chi_j = \chi_i$ and $g_j = g_i$. Now, as $k \not\sim j$:

$$1 = \chi_j(g_k) \chi_k(g_j) = \chi_i(g_k) \chi_k(g_i),$$

which contradicts the fact that $k \sim i$. Hence $s_{ij} = 0$. If, on the contrary, both x_i and x_j determine a one-dimensional connected component and $s_{ij} \neq 0$, then $j \in \mathbb{I}(i)$ implies $q_{ii} = q_{ij} = q_{ji} = q_{jj} = -1$. \square

We recall from p. 8 that $\mathfrak{u}_{|i}(\boldsymbol{\lambda}) \subseteq \mathfrak{u}(\boldsymbol{\lambda})$ stands for the subalgebra generated by V_i and H . The following is straightforward.

Lemma 5.5. *Let $\psi \in \text{Isom}(\mathfrak{u}(\boldsymbol{\lambda}), \mathfrak{u}(\boldsymbol{\lambda}'))$. Then there is $\tau \in \mathbb{S}_m$ such that $T_i := T|_{V_i} : V_i \rightarrow V'_{\tau(i)}$ is an isomorphism of braided vector spaces for each $i \in \mathbb{I}_m$. Moreover,*

$$\psi_{|i} := \psi|_{\mathfrak{u}_{|i}(\boldsymbol{\lambda})} : \mathfrak{u}_{|i}(\boldsymbol{\lambda}) \rightarrow \mathfrak{u}_{|\tau(i)}(\boldsymbol{\lambda}')$$

is a Hopf algebra isomorphism. \square

By Lemma 4.1, $\mathfrak{u}_{|i}(\boldsymbol{\lambda}) \simeq \mathfrak{u}(\boldsymbol{\lambda}_{|i})$ and hence each $\psi \in \text{Isom}(\mathfrak{u}(\boldsymbol{\lambda}), \mathfrak{u}(\boldsymbol{\lambda}'))$ induces isomorphisms $\mathfrak{u}(\boldsymbol{\lambda}_{|i}) \rightarrow \mathfrak{u}(\boldsymbol{\lambda}'_{|\tau(i)})$ for all $i \in \mathbb{I}_m$. We denote by

$$\text{Isom}(\boldsymbol{\lambda}_{|1}, \boldsymbol{\lambda}'_{|\tau(1)}) \times_H \cdots \times_H \text{Isom}(\boldsymbol{\lambda}_{|m}, \boldsymbol{\lambda}'_{|\tau(m)}) \subset \bigtimes_{i=1}^m \text{Isom}(\boldsymbol{\lambda}_{|i}, \boldsymbol{\lambda}'_{|\tau(i)})$$

the subset of all m -tuples $((\varphi_i, \sigma_i, s_i))_{i \in \mathbb{I}_m}$ with $\varphi_i = \varphi_j$, $i, j \in \mathbb{I}_m$. Therefore, Lemma 5.5 defines a map

$$(5.4) \quad \text{Res} : \text{Isom}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \longrightarrow \text{Isom}(\boldsymbol{\lambda}_{|1}, \boldsymbol{\lambda}'_{|\tau(1)}) \times_H \cdots \times_H \text{Isom}(\boldsymbol{\lambda}_{|m}, \boldsymbol{\lambda}'_{|\tau(m)})$$

Lemma 5.6. *The map Res is injective.*

Proof. Let $(\varphi, \sigma, s) \in \text{Isom}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ and denote by $\psi : \mathfrak{u}(\boldsymbol{\lambda}) \rightarrow \mathfrak{u}(\boldsymbol{\lambda}')$ the corresponding Hopf algebra isomorphism from Theorem 5.3. The collection of maps $(\psi_{|i})_{i \in \mathbb{I}_m}$ controls the behavior of ψ when restricted to the first term of the coradical filtration of $\mathfrak{u}(\boldsymbol{\lambda})$ and hence determines the morphism ψ . \square

To describe the image of the map Res, we set, for each $i \in \mathbb{I}_m$,

$$\text{Isom}_0(\boldsymbol{\lambda}_{|i}, \boldsymbol{\lambda}'_{|\tau(i)}) = \{(\varphi, \sigma, s) \in \text{Isom}(\boldsymbol{\lambda}_{|i}, \boldsymbol{\lambda}'_{|\tau(i)}) \mid \lambda'_0 = s \cdot \lambda_0^{\sigma_i}\}$$

and define $\text{Isom}_0(\boldsymbol{\lambda}_{|1}, \boldsymbol{\lambda}'_{|\tau(1)}) \times_H \cdots \times_H \text{Isom}_0(\boldsymbol{\lambda}_{|m}, \boldsymbol{\lambda}'_{|\tau(m)})$ accordingly. By definition, the image of Res lies inside this subset.

Proposition 5.7. *The map Res defines a bijective correspondence*

$$(5.5) \quad \text{Isom}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \xrightarrow{\sim} \text{Isom}_0(\boldsymbol{\lambda}_{|1}, \boldsymbol{\lambda}'_{|\tau(1)}) \times_H \cdots \times_H \text{Isom}_0(\boldsymbol{\lambda}_{|m}, \boldsymbol{\lambda}'_{|\tau(m)}).$$

Proof. First, Res is injective by Lemma 5.6. Now, let $((\varphi, \sigma_i, s_i))_{i \in \mathbb{I}_m}$ belong to the right hand side of (5.5), for some $\varphi \in \text{Isom}(H, H')$. Then, via the natural identification $\mathbb{I}_{\theta_1} \times \cdots \times \mathbb{I}_{\theta_m} = \mathbb{I}_{\theta}$:

$$(a) \quad \sigma = \sigma_1 \times \cdots \times \sigma_m \in \mathbb{S}_{\mathbf{q}}.$$

$$(b) \quad \text{The block matrix } s = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & s_m \end{bmatrix} \text{ defines an element in } \mathbf{L}.$$

Indeed, (a) is clear from the definition and (b) follows by Remark 5.4. Hence, $(\varphi, \sigma, s) \in \text{Isom}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ and it is mapped onto $((\varphi, \sigma_i, s_i))_{i \in \mathbb{I}_m}$ by Res. \square

APPENDIX: NONZERO CLEFT OBJECTS

In this section we show that each algebra $\tilde{\mathcal{E}}_{\mathbf{q}} = \tilde{\mathcal{E}}_{\mathbf{q}}(\boldsymbol{\lambda})$ is nonzero for each connected matrix \mathbf{q} and each $\boldsymbol{\lambda} \in \mathcal{R}$, see (1.3). This completes the proof of Proposition 3.8.

5.1. Tools and techniques. For each \mathbf{q} we show that $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$ using a combination of the following results and resources.

5.1.1. For diagrams of small rank we explicitly compute the Gröbner basis of the defining ideal of $\tilde{\mathcal{E}}_{\mathbf{q}}$. This is achieved using computer program [GAP], together with the package **GBNP** by [CG]. The corresponding `log` files are stored in the authors' webpages.

5.1.2. We invoke some results from previous articles. The case in which only the powers of simple roots are deformed follows from [A+, Lemma 5.16]. On the other hand, the nontrivial deformations of the quantum Serre relations was investigated in [AG, Proposition 3.2].

5.1.3. Assume that all the generalized quantum Serre relations involving two letters $y_i, y_j, i < j \in \mathbb{I}$ and $q_{ij}q_{ji} \neq 1$ (i.e. adjacent vertices), are not deformed. Let $\mathcal{G}[ij] \subset \mathcal{G}$ be the subset containing these relations.

First, assume further that there is no $k \neq i, j \in \mathbb{I}$ adjacent to both i and j . Let $\mathbf{q}' = (q'_{st})_{s,t \in \mathbb{I}}$ be the matrix such that $q'_{st} = q_{st}$ when $(s, t) \neq (j, i)$ and $q'_{ji} = q_{ij}^{-1}$. Hence \mathbf{q}' has two connected components arising by cutting the original diagram between i and j . Thus the set of defining relations of $\mathfrak{B}_{\mathbf{q}'}$ is $\mathcal{G}' = (\mathcal{G} \setminus \mathcal{G}[ij]) \cup \{x_{ij}\}$. Now $\tilde{\mathcal{E}}_{\mathbf{q}}(\boldsymbol{\lambda})/\langle y_{ij} \rangle$ projects onto $\tilde{\mathcal{E}}_{\mathbf{q}'}(\boldsymbol{\lambda}')$ for $\boldsymbol{\lambda}' = (\lambda'_r)_{r \in \mathcal{G}'}$ with $\lambda'_r = \lambda_r$ if $r \in \mathcal{G} \setminus \mathcal{G}[ij]$ and $\lambda'_{ij} = 0$. In this case we show that $\tilde{\mathcal{E}}_{\mathbf{q}}(\boldsymbol{\lambda}) \neq 0$ using a recursive argument.

Second, assume $i < j < k \in \mathbb{I}$ are pairwise adjacent and all the generalized quantum Serre relations involving either the letters y_i and y_j or y_i and y_k are not deformed. In this case, we proceed in a similar way by cutting the diagram between i and j , and also between i and k .

5.1.4. Assume that all the relations involving a letter $y_i, i \in \mathbb{I}$, are not deformed. Let $\mathcal{G}[i] \subset \mathcal{G}$ be the subset containing these relations. Let \mathbf{q}' be the submatrix of \mathbf{q} obtained by removing the i -th row and column. Now the set of relations of $\mathfrak{B}_{\mathbf{q}'}$ is $\mathcal{G}' = \mathcal{G} \setminus \mathcal{G}[i]$ and $\tilde{\mathcal{E}}_{\mathbf{q}}(\boldsymbol{\lambda})/\langle y_i \rangle$ projects onto $\tilde{\mathcal{E}}_{\mathbf{q}'}(\boldsymbol{\lambda}')$, $\boldsymbol{\lambda}' = (\lambda_r)_{r \in \mathcal{G}'}$. Again, a recursive argument shows that $\tilde{\mathcal{E}}_{\mathbf{q}}(\boldsymbol{\lambda}) \neq 0$.

5.1.5. In many cases, we can restrict to the case in which a single relation $r \in \mathcal{G}$ is deformed. Then an iterative application of Lemma 3.3 shows that $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$ by deforming one relation at a time. In each case, we can restrict to the subdiagram involving just the letters in the relation, using the technique described in §5.1.4.

6. CARTAN TYPE

Conventions. In what follows $q \in \mathbb{k}^\times$ has order $N > 1$. All generalized Dynkin diagrams are numbered from the left to the right and from bottom to top. If a numbered display contains several equalities (or diagrams), they will be referred to with roman letters from the left to the right.

6.1. Type A_θ , $\theta \geq 1$. For $N > 2$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$y_{ij} = 0, \quad i < j - 1; \quad y_{ij} = \lambda_{ij}, \quad |j - i| = 1.$$

This algebra is nonzero, see [AAG, Proposition 5.2].

If $N = 2$, then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$y_{ij} = \lambda_{ij}, \quad i < j - 1; \quad [y_{(i+2)}, y_{i+1}]_c = \nu_i.$$

This algebra is nonzero, see [AAG, Proposition 4.2].

6.2. Type B_θ , $\theta \geq 2$, $N > 2$.

$N > 4$. The algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$y_{ij} = 0, \quad i < j - 1; \quad y_{ij} = \lambda_{ij}, \quad i < \theta, |j - i| = 1; \quad y_{\theta\theta\theta\theta-1} = \lambda_{\theta\theta\theta\theta-1}.$$

If $N > 5$, then $\lambda_{\theta\theta\theta\theta-1} = \lambda_{ij} = 0$ for all i, j , so $\tilde{\mathcal{E}}_q = \tilde{\mathfrak{B}}_q$. For $N = 5$ all these scalars are 0 again, unless $q = \begin{pmatrix} q^2 & q \\ q^2 & q \end{pmatrix}$, in which case $\lambda_{\theta-1\theta-1\theta}$, $\lambda_{\theta\theta\theta\theta-1}$ can be non-zero. This algebra projects onto the corresponding algebra of type B_2 , which is nonzero, see **b2.log**.

$N = 4$. The algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$y_{ij} = \lambda_{ij}, \quad i < j - 1; \quad [y_{(i+2)}, y_{i+1}]_c = \nu_i, \quad i < \theta; \quad y_{\theta\theta\theta\theta-1} = 0.$$

Notice that $\nu_{\theta-1} = 0$ since $\chi_{\theta-2}\chi_{\theta-1}^2\chi_\theta(g_{\theta-2}g_{\theta-1}^2g_\theta) = -q$. Also, $\lambda_{i\theta} = 0$ for all $i \leq \theta - 2$, so $\tilde{\mathcal{E}}_q$ projects over $\tilde{\mathcal{E}}_{q'}$, q' of type $A_{\theta-1}$.

$N = 3$. The algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{ij} &= 0, \quad i + 1 < j < \theta; & y_{ij} &= \lambda_{ij}, \quad i < \theta, |j - i| = 1; \\ y_{i\theta} &= \lambda_{i\theta}, \quad i + 1 < \theta; & [y_{\theta\theta\theta-1\theta-2}, y_{\theta\theta-1}]_c &= 0. \end{aligned}$$

Notice that $\lambda_{\theta-1\theta-1\theta} = 0$. If $\lambda_{\theta-1\theta-1\theta-2} = \lambda_{\theta-2\theta-2\theta-1} = 0$, then $\tilde{\mathcal{E}}_q$ projects onto $\tilde{\mathcal{E}}_{q'}$, q' of type $A_{\theta-2} \times A_1$. If either $\lambda_{\theta-1\theta-1\theta-2} \neq 0$ or $\lambda_{\theta-2\theta-2\theta-1} \neq 0$, then $\lambda_{\theta-2\theta-2\theta-3} = \lambda_{\theta-3\theta-3\theta-2} = 0$; $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta-3} \rangle$ projects onto $\tilde{\mathcal{E}}_{q'}$, q' of type $A_{\theta-3} \times B_3$, which is nonzero, see **b3.log**.

6.3. Type C_θ , $\theta \geq 3$, $N > 2$.

$N > 3$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_{\theta}$, with defining relations

$$\begin{aligned} y_{ij} &= 0, \quad i < j - 1; & y_{iij} &= 0, \quad i \leq \theta - 2, |j - i| = 1; \\ y_{\theta-1\theta-1\theta-1\theta} &= \lambda_{\theta-1\theta-1\theta-1\theta}; & y_{\theta\theta\theta-1} &= \lambda_{\theta\theta\theta-1}. \end{aligned}$$

If $N \neq 5$, then $\tilde{\mathcal{E}}_{\mathbf{q}} = \tilde{\mathfrak{B}}_{\mathbf{q}}$. For $N = 5$, $\lambda = 0$ again, unless $\mathbf{q} = \begin{pmatrix} q & q^2 \\ q & q^2 \end{pmatrix}$, in which case $\lambda_{\theta-1\theta-1\theta-1\theta}, \lambda_{\theta\theta\theta-1}$ can be non-zero. Thus $\tilde{\mathcal{E}}_{\mathbf{q}}$ projects onto the corresponding algebra of type B_2 , which is nonzero, see `b2.log`.

$N = 3$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_{\theta}$, with defining relations

$$\begin{aligned} y_{ij} &= 0, \quad i + 1 < j < \theta; & y_{iij} &= \lambda_{iij}, \quad j = i \pm 1, i < \theta - 1, & y_{\theta\theta\theta-1} &= 0; \\ y_{i\theta} &= \lambda_{i\theta}, \quad i < \theta - 1; & [[y_{(\theta-2\theta)}, y_{\theta-1}]_c, y_{\theta-1}]_c &= \lambda_{(\theta-2\theta)}. \end{aligned}$$

If $\lambda_{(\theta-2\theta)} = 0$, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' of type $A_{\theta-1} \times A_1$. If $\lambda_{(\theta-2\theta)} \neq 0$, then we may restrict to rank 3 following §5.1.5 and $\mathbf{q} = \begin{pmatrix} \zeta & b\zeta^2 & b^{-3}\zeta^2 \\ b^{-1} & \zeta & b \\ b^3\zeta & b^{-1}\zeta & \zeta^2 \end{pmatrix}$; this algebra is not zero by `c3.log`.

6.4. Type D_{θ} , $\theta \geq 4$.

$N > 2$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_{\theta}$, with defining relations

$$\begin{aligned} y_{\theta-1\theta} &= 0; & y_{ij} &= 0, \quad i < j - 1, (i, j) \neq (\theta - 2, \theta); & y_{\theta\theta\theta-2} &= \lambda_{\theta\theta\theta-2}; \\ y_{iij} &= \lambda_{iij}, & |j - i| &= 1, i, j \neq \theta; & y_{\theta-2\theta-2\theta} &= \lambda_{\theta-2\theta-2\theta}. \end{aligned}$$

If $N \neq 3$, then $\lambda = 0$, so $\tilde{\mathcal{E}}_{\mathbf{q}} = \tilde{\mathfrak{B}}_{\mathbf{q}}$. If $N = 3$, then either $\lambda_{\theta-2\theta-2\theta-1} = \lambda_{\theta-1\theta-1\theta-2} = 0$ or else $\lambda_{\theta-2\theta-2\theta} = \lambda_{\theta\theta\theta-2} = 0$ by [AAG, Lemma 5.1 (2)]; we may assume the last case, and $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{\theta} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' of type $A_{\theta-1}$.

$N = 2$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_{\theta}$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, \quad i < j - 1, (i, j) \neq (\theta - 2, \theta); & y_{\theta-1\theta} &= \lambda_{\theta-1\theta}; \\ [y_{(ii+2)}, y_{i+1}]_c &= \nu_i, \quad i \leq \theta - 3; & [y_{\theta-3\theta-2\theta}, y_{\theta-2}]_c &= \nu'_{\theta-2}. \end{aligned}$$

By Diamond Lemma [B] $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$: the defining relations are as for type A , so the proof of [AAG, Proposition 4.2] applies to solve the ambiguities.

6.5. Type E_{θ} , $6 \leq \theta \leq 8$. The proof is analogous to the case D_{θ} .

6.6. Type F_4 , $N > 2$.

$N > 4$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{ij} &= 0, \quad i < j - 1; & y_{112} &= 0, & y_{221} &= 0, & y_{334} &= 0, \\ y_{443} &= 0, & y_{223} &= \lambda_{223}, & y_{3332} &= \lambda_{3332}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_1, y_4 \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' of type B_2 .

$N = 4$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{ij} &= 0, & i < j - 1; & & [y_{(13)}, y_2]_c &= 0; & & y_{3332} &= 0 \\ y_{334} &= 0; & & & y_{443} &= 0. \end{aligned}$$

Hence, $\tilde{\mathcal{E}}_{\mathbf{q}} = \tilde{\mathfrak{B}}_{\mathbf{q}} \neq 0$.

$N = 3$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, & i < j - 1; & & [[y_{(24)}, y_3]_c, y_3]_c &= \lambda_{(24)}; \\ y_{iij} &= \lambda_{iij}, & j = i \pm 1, (i, j) \neq (3, 2); & & [y_{3321}, y_{32}]_c &= \lambda_{321}. \end{aligned}$$

Notice that $\lambda_{223} = 0$. If $\lambda_{321} = \lambda_{(24)} = 0$, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' of type $A_2 \times A_2$. If $\lambda_{321} \neq 0$ but all the other λ 's are 0, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' of type $B_3 \times A_1$. If $\lambda_{(24)} \neq 0$ but all the other λ 's are 0, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' of type $C_3 \times A_1$. The general case follows by applying Lemma 3.3 twice.

6.7. Type G_2 , $N > 3$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_1, y_2 with defining relations

$$y_{112} = \lambda_1; \quad y_{22221} = \lambda_2$$

with $\lambda_1 = \lambda_2 = 0$ unless $N = 7$ and $\mathbf{q} = \begin{pmatrix} q & q^3 \\ q & q^3 \end{pmatrix}$. This algebra is nonzero, see `g2.log`.

7. STANDARD TYPE

7.1. Standard type B_{θ} , $\theta \geq 2$, $\zeta \in G'_3$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_{\theta}$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, & i < j - 1; & & [y_{(i-1i+1)}, y_i]_c &= \nu_i, & q_{ii} &= -1; \\ y_{iii\pm 1} &= 0, & q_{ii} &= -\zeta^{\pm 1}; & [y_{\theta\theta\theta-1\theta-2}, y_{\theta\theta-1}]_c &= 0; \\ y_{\theta}^3 &= \mu_{\theta}; & & & [y_{\theta\theta\theta-1}, y_{\theta\theta-1}]_c &= \lambda_{\theta\theta-1}, & q_{\theta-1\theta-1} &= -1; \\ y_i^2 &= \mu_i, & q_{ii} &= -1. \end{aligned}$$

As $\chi_{\theta-1}\chi_{\theta}^2\chi_{\theta+1}(g_{\theta-1}g_{\theta}^2g_{\theta+1}) \neq 1$, $\nu_{\theta-1} = 0$. Hence, if either $q_{\theta-1\theta-1} \neq -1$ or $q_{\theta-1\theta-1} = -1$ but $\lambda_{\theta\theta-1} = 0$, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of super type A and another of type A_1 .

If $q_{\theta-1\theta-1} = -1$ and $\lambda_{\theta\theta-1} \neq 0$, then either $q_{\theta-2\theta-2} \neq 1$ or else $q_{\theta-2\theta-2} \neq 1$, $\nu_{\theta-2} = -1$. In both cases, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of super type A and another of type B_2 with matrix $\begin{pmatrix} -1 & \zeta \\ -1 & \zeta \end{pmatrix}$; this algebra is nonzero, see `b2st.log`.

7.2. **Standard type** G_2 , $\zeta \in G'_8$. The Weyl groupoid has three objects.

For $\begin{smallmatrix} \zeta^2 & \zeta \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} \zeta^2 & \zeta^{-1} \\ \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_1, y_2 with defining relations

$$y_1^4 = \lambda_1; \quad y_{221} = \lambda_2; \quad [y_{1112}, y_{112}] = \lambda_3 + 2\lambda_2(1+q)^2 y_2 y_{12},$$

with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ unless $\mathbf{q} = \begin{pmatrix} \zeta^2 & \zeta^{-1} \\ \zeta^2 & \zeta^{-1} \end{pmatrix}$; see `g2-st-a.log` for the deformation of the generalized quantum Serre relation. This algebra is nonzero, see `g2-st-a.log`.

For $\begin{smallmatrix} \zeta^2 & \zeta^3 & -1 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_1, y_2 with defining relations

$$y_1^4 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [y_1, y_{3\alpha_1+2\alpha_2}]_c + \frac{q_{12}}{1-\zeta} y_{112}^2 = 0,$$

with $\lambda_1 \lambda_2 = 0$. This algebra is nonzero, using [A+, Lemma 5.16]. The last relation is not primitive: it remains undeformed by direct computation.

For $\begin{smallmatrix} \zeta & \zeta^5 & -1 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_1, y_2 with defining relations

$$y_{11112} = \lambda_1; \quad y_2^2 = \lambda_2, \quad [y_{112}, y_{12}]_c, [y_{12}]_c = \lambda_3;$$

with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ unless $\mathbf{q} = \begin{pmatrix} \zeta & -1 \\ \zeta & -1 \end{pmatrix}$. This algebra is nonzero, see `g2-st-c.log`.

8. SUPER TYPE

8.1. **Type A** $(j-1|\theta-j)$, $\theta \geq 1$, $1 \leq j \leq \theta$, $N > 2$. The algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_{\theta}$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, \quad i < j-1; & y_{iii\pm 1} &= \lambda_{iii\pm 1}, \quad q_{ii} \neq -1; \\ y_i^2 &= 0, \quad q_{ii} = -1; & [y_{(i-1i+1)}, y_i]_c &= \nu_i, \quad q_{ii} = -1. \end{aligned}$$

By induction on θ . If $\theta = 2$, then there are 3 diagrams.

$\begin{smallmatrix} q & q^{-1} \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} q & q^{-1} \\ \circ & \circ \end{smallmatrix}$: It is of Cartan type, already solved.

$\begin{smallmatrix} -1 & q & -1 \\ \circ & \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} -1 & q & -1 \\ \circ & \circ & \circ \end{smallmatrix}$: $y_i^2 = \mu_i$, $i = 1, 2$. Hence [A+, Lemma 5.16] applies.

$\begin{smallmatrix} q & q^{-1} & -1 \\ \circ & \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} q & q^{-1} & -1 \\ \circ & \circ & \circ \end{smallmatrix}$: $y_{112} = \lambda_{112}$, $y_2^2 = \mu_2$. If $\lambda_{112} = 0$, then [A+, Lemma 5.16]

applies. Otherwise, $\mathbf{q} = \begin{pmatrix} q & -1 \\ q & -1 \end{pmatrix}$, $q \in G'_4$; $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$ by `a2-super.log`.

For $\theta = 3$, we have six cases:

$\begin{smallmatrix} q & q^{-1} & q & q^{-1} \\ \circ & \circ & \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} q & q^{-1} & q & q^{-1} \\ \circ & \circ & \circ & \circ \end{smallmatrix}$: It is of Cartan type, already solved.

$\begin{smallmatrix} q & q^{-1} \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} q & q^{-1} \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$: If $\lambda_{223} = 0$, then we quotient by y_{23} . If $\lambda_{223} \neq 0$, then

$q \in G'_4$, so $\lambda_{112} = \lambda_{221} = 0$ and we quotient by y_{12} .

$\begin{smallmatrix} q & q^{-1} \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} -1 & q \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} q^{-1} \\ \circ \end{smallmatrix}$: First assume $\nu_2 = 0$. If $\lambda_{112}\lambda_{332} = 0$, then it projects

over a rank 2 case. Otherwise, $\mathfrak{q} = \begin{pmatrix} q & -1 & b \\ q & -1 & -q \\ -b & -1 & -q \end{pmatrix}$, $b = \pm q \in G'_4$; this algebra

is nonzero, see **a3-super-1.log**. If $\nu_2 \neq 0$, then $\mathfrak{q} = \begin{pmatrix} q & a & q^{-1}a^{-2} \\ q^{-1}a^{-1} & -1 & qa \\ qa^2 & a^{-1} & q^{-1} \end{pmatrix}$, $a \neq 0$; this algebra is nonzero, see **a3-super-2.log**. The general case follows by Lemma 3.3.

$\begin{smallmatrix} -1 & q \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} -1 & q^{-1} \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} q \\ \circ \end{smallmatrix}$: as $\nu_2 = 0$, we quotient by y_{12} .

$\begin{smallmatrix} -1 & q^{-1} \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} q & q^{-1} \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$: If $\lambda_{221}\lambda_{223} = 0$, then we quotient either by y_{12} or

y_{23} . Otherwise, $\mathfrak{q} = \begin{pmatrix} -1 & q & -1 \\ -1 & q & -1 \\ -1 & q & -1 \end{pmatrix}$, $q \in G'_4$; this algebra is nonzero, see

a3-super-3.log.

$\begin{smallmatrix} -1 & q^{-1} \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} -1 & q \\ \circ & \circ \end{smallmatrix} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$: If $\nu_2 = 0$, then we quotient by y_{12} . If $\nu_2 \neq 0$, then $\mathfrak{q} =$

$\begin{pmatrix} -1 & a & -a^{-2} \\ q^{-1}a^{-1} & -1 & qa \\ -a^2 & a^{-1} & -1 \end{pmatrix}$, $a \neq 0$; this algebra is nonzero, see **a3-super-4.log**.

For the inductive step, if the λ 's involving $y_\theta, y_{\theta+1}$ are 0, then $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{\theta\theta+1} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, of super type A and A_1 . If the λ 's involving y_θ and $y_{\theta+1}$ are the unique non-zero scalars, then it projects over the rank 2 or rank 3 cases. The general case follows by Lemma 3.3.

8.2. Type $\mathbf{B}(m|n)$, $q \notin G_4$.

$N > 4$. The algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, i < j - 1; & y_i^2 &= \mu_i, q_{ii} = -1; & y_{\theta\theta\theta\theta-1} &= \lambda_{\theta\theta\theta\theta-1}; \\ y_{iij} &= \lambda_{iij}, i < \theta, |j - i| = 1, q_{ii} \neq -1; & [y_{(i-1)i+1}, y_i]_c &= \nu_i, q_{ii} = -1. \end{aligned}$$

First assume $q_{\theta-1\theta-1} \neq -1$. If $\lambda_{\theta\theta\theta\theta-1} = \lambda_{\theta-1\theta-1\theta} = 0$, then we consider the quotient $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{\theta-1\theta} \rangle$ following §5.1.3. If either $\lambda_{\theta\theta\theta\theta-1} \neq 0$ or $\lambda_{\theta-1\theta-1\theta} \neq 0$, then $q \in G'_5$ and $\nu_{\theta-2} = 0$; hence $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of super type A and another of Cartan type B_2 .

When $q_{\theta-1\theta-1} = -1$, $\nu_{\theta-1} = 0$ since $\chi_{\theta-2}\chi_{\theta-1}^2\chi_\theta \neq \epsilon$. If $\lambda_{\theta\theta\theta\theta-1} = 0$, then again we consider $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$. If $\lambda_{\theta\theta\theta\theta-1} \neq 0$, then $q \in G'_6$ and $\nu_{\theta-2} = 0$; hence $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of super type A and another of Cartan type B_2 with matrix $\begin{pmatrix} -1 & q \\ -1 & q \end{pmatrix}$; this algebra is nonzero, see **b2supera.log**.

$N = 3$. The algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, \quad i < j - 1; & [y_{(i-1)i+1}, y_i]_c &= \nu_i, \quad q_{ii} = -1; \\ y_{iii\pm 1} &= \lambda_{iii\pm 1}, \quad q_{ii} \neq -1; & [y_{\theta\theta\theta-1}, y_{\theta\theta-1}]_c &= \lambda_{\theta\theta-1}, \quad q_{\theta-1\theta-1} = -1; \\ [y_{\theta\theta\theta-1\theta-2}, y_{\theta\theta-1}]_c &= \lambda_{\theta\theta-1\theta-2}; & y_i^2 &= \mu_i, \quad q_{ii} = -1. \end{aligned}$$

First assume $\lambda_{\theta\theta-1\theta-2} \neq 0$. Then $q_{ii} = q^2$ for $i = \theta - 2, \theta - 1$. Moreover, $\lambda_{\theta-2\theta-2\theta-3} = 0, \nu_{\theta-3} = 0$ if $q_{\theta-3\theta-3} = -1$, $\lambda_{\theta-3\theta-3\theta-2} = 0$ if $q_{\theta-3\theta-3} = q^2$; hence $\tilde{\mathcal{E}}_q / \langle y_{\theta-3\theta-2} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type A and another of Cartan type B_3 .

Now assume $\lambda_{\theta\theta-1\theta-2} = 0$. If either $q_{\theta-1\theta-1} = q^2$ or $q_{\theta-1\theta-1} = -1$, $\lambda_{\theta\theta-1} = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type A and another of type A_1 . If $q_{\theta-1\theta-1} = -1, \lambda_{\theta\theta-1} \neq 0$, then $\nu_{\theta-1} = 0, \nu_{\theta-2} = 0$ if $q_{\theta-2\theta-2} = -1, \lambda_{\theta-2\theta-2\theta-1} = 0$ if $q_{\theta-2\theta-2} \neq -1$; hence $\tilde{\mathcal{E}}_q / \langle y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type A and another of Cartan type B_2 with matrix $\begin{pmatrix} -1 & \zeta \\ 1 & \zeta \end{pmatrix}$; this algebra is nonzero, see `b2superb.log`.

8.3. Type $\mathbf{D}(m|n), N > 2$. The Weyl groupoid has objects with generalized Dynkin diagrams of six possible shapes.

For the diagram $\mathbf{A}_{\theta-2}(q; i_1, \dots, i_j) \xrightarrow{q^{-1}} \circ \xrightarrow{q} \xrightarrow{q^{-2}} \circ \xrightarrow{q^2}$, there are 3 cases:

$\diamond N > 4$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{iii\pm 1} &= 0, \quad i < \theta - 1, q_{ii} \neq -1; \\ y_{\theta-1\theta-1\theta-2} &= 0; & y_{ij} &= \lambda_{ij}, \quad i < j - 1; \\ y_{\theta\theta\theta-1} &= \lambda_{\theta\theta\theta-1}; & [y_{(i-1)i+1}, y_i]_c &= \nu_i, \quad q_{ii} = -1, 2 \leq i \leq \theta - 2; \\ y_{\theta-1\theta-1\theta-1\theta} &= \lambda_{\theta-1\theta-1\theta-1\theta}; & y_i^2 &= \mu_i, \quad q_{ii} = -1. \end{aligned}$$

If $\lambda_{\theta-1\theta-1\theta-1\theta} = \lambda_{\theta\theta\theta-1} = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type A and another of Cartan type A_1 . If either $\lambda_{\theta-1\theta-1\theta-1\theta} \neq 0$ or $\lambda_{\theta\theta\theta-1} \neq 0$, then $\nu_{\theta-2} = 0$ if $q_{\theta-2\theta-2} = -1$; hence $\tilde{\mathcal{E}}_q / \langle y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type A and another of Cartan type B_2 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{iii\pm 1} &= \lambda_{iii\pm 1}, \quad i < \theta - 1, q_{ii} \neq -1; \\ y_{\theta-1\theta-1\theta-2} &= \lambda_{\theta-1\theta-1\theta-2}; & y_{ij} &= \lambda_{ij}, \quad i < j - 1; \\ y_{\theta-1\theta-1\theta-1\theta} &= 0; & [y_{(i-1)i+1}, y_i]_c &= \nu_i, \quad q_{ii} = -1, i \neq \theta - 1; \\ [y_{(\theta-2\theta)}, y_{\theta-1\theta}]_c &= 0; & y_i^2 &= \mu_i, \quad q_{ii} = -1. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type A and another of Cartan type A_1 .

$\diamond N = 3$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{iii\pm 1} &= \lambda_{iii\pm 1}, \quad i < \theta - 1, q_{ii} \neq -1; \\ y_{\theta-1\theta-1\theta-2} &= \lambda_{\theta-1\theta-1\theta-2}; & y_{ij} &= \lambda_{ij}, \quad i < j - 1; \end{aligned}$$

$$\begin{aligned} y_{\theta\theta\theta-1} &= 0; & [y_{(i-1i+1)}, y_i]_c &= \nu_i, q_{ii} = -1, i \neq \theta - 1; \\ y_i^2 &= \mu_i, q_{ii} = -1; & [[y_{(\theta-2\theta)}, y_{\theta-1}]_c, y_{\theta-1}]_c &= \lambda_{(\theta-2\theta)}. \end{aligned}$$

Here, $q_{\theta-2\theta-2} = q$ if $\lambda_{(\theta-2\theta)} \neq 0$; hence we work as for Cartan type C_θ .

For the diagram $\mathbf{A}_{\theta-2}(q; i_1, \dots, i_j) \xrightarrow{q} \circ \xrightarrow{-1} \circ \xrightarrow{q^{-2}} \circ \xrightarrow{q^2} \circ$, there are 2 cases:

$\diamond N \neq 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_i^2 &= \mu_i, q_{ii} = -1; & y_{iii\pm 1} &= 0, i < \theta - 1, q_{ii} \neq -1; \\ y_{ij} &= \lambda_{ij}, i < j - 1; & [[y_{\theta-2\theta-1}, y_{(\theta-2\theta)}]_c, y_{\theta-1}]_c &= 0, q_{\theta-2\theta-2} = -1; \\ & & [[[y_{(\theta-3\theta)}, y_{\theta-1}]_c, y_{\theta-2}]_c, y_{\theta-1}]_c &= 0, q_{\theta-2\theta-2} \neq -1; \\ y_{\theta\theta\theta-1} &= \lambda_{\theta\theta\theta-1}; & [y_{(i-1i+1)}, y_i]_c &= \nu_i, q_{ii} = -1, i \neq \theta - 1. \end{aligned}$$

If $\lambda_{\theta\theta\theta-1} = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathbf{q}'$ with two components, one of super type A and another of type A_1 . If $\lambda_{\theta\theta\theta-1} \neq 0$, then either $q_{\theta-2\theta-2} \neq -1$, or else $q_{\theta-2\theta-2} = -1$, $\nu_{\theta-2} = 0$; in both cases $\tilde{\mathcal{E}}_q / \langle y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathbf{q}'$ with two components, one of super type $A_{\theta-2}$ and another of super type A_2 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, i < j - 1; & y_{iii\pm 1} &= \lambda_{iii\pm 1}, i < \theta - 1, q_{ii} \neq -1; \\ [[y_{\theta-2\theta-1}, y_{(\theta-2\theta)}]_c, y_{\theta-1}]_c &= \lambda_{(\theta-2\theta)}, q_{\theta-2\theta-2} = -1; \\ y_{\theta-1\theta}^2 &= \mu_{\theta-1\theta}; & [[[y_{(\theta-3\theta)}, y_{\theta-1}]_c, y_{\theta-2}]_c, y_{\theta-1}]_c &= 0, q_{\theta-2\theta-2} \neq -1; \\ y_i^2 &= \mu_i, q_{ii} = -1; & [y_{(i-1i+1)}, y_i]_c &= \nu_i, q_{ii} = -1, i \neq \theta - 1. \end{aligned}$$

First consider $q_{\theta-2\theta-2} \neq -1$. If $\lambda_{iii\pm 1} \neq 0$, $i = \theta - 2$, $\mu_{\theta-1\theta} \neq 0$, then $\tilde{\mathcal{E}}_q / \langle y_{\theta-4\theta-3} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathbf{q}'$ with two components, one of super type $A_{\theta-4}$ and another with matrix

$\begin{pmatrix} -1 & q & -1 & 1 \\ -1 & q & -1 & \pm 1 \\ -1 & q & -1 & -1 \\ 1 & \pm 1 & 1 & -1 \end{pmatrix}$; this algebra is not zero, see `CDrk4a.log`.

If $\mu_{\theta-1\theta} = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathbf{q}'$ with two components, one of super type $A_{\theta-1}$ and another of type A_1 . If $\lambda_{\theta-2\theta-2\theta-1} = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathbf{q}'$ with two components, one of super type $A_{\theta-2}$ and another of Cartan type A_2 . If $\lambda_{\theta-2\theta-2\theta-3} = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{\theta-3\theta-2} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathbf{q}'$ with two components, one of super type $A_{\theta-3}$ and the other is a subalgebra of `CDrk4a.log`.

Now fix $q_{\theta-2\theta-2} = -1$. A similar analysis reduces to the case $\nu_{\theta-2} \neq 0$, $\lambda_{(\theta-2\theta)} \neq 0$; here, $\tilde{\mathcal{E}}_q / \langle y_{\theta-4\theta-3} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathbf{q}'$ with two components, one of super

type $A_{\theta-4}$ and another with matrix $\begin{pmatrix} -1 & \bar{q}a^{-1} & a^2 & -a^{-4} \\ a & -1 & a^{-1} & a^3 \\ a^{-2} & qa & -1 & a^{-2} \\ -a^4 & a^{-3} & -a^2 & -1 \end{pmatrix}$, $a \neq 0$; this algebra

is not zero, see `CDrk4b.log`.

For the diagram

$$\begin{array}{ccc} & \overset{-1}{\circ} & \\ & \downarrow q^{-1} & \searrow q^2 \\ \mathbf{A}_{\theta-3}(q; i_1, \dots, i_j) & \xrightarrow[q^{-1}]{q} \overset{-1}{\circ} & \xrightarrow[q^{-1}]{-1} \overset{-1}{\circ} \end{array} \quad \text{there are 2 cases:}$$

$\diamond N \neq 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{iii\pm 1} &= 0, \quad i < \theta - 1, q_{ii} \neq -1; \\ y_{\theta-2\theta-2\theta} &= 0; \quad [y_{(i-1i+1)}, y_i]_c = \nu_i, \quad q_{ii} = -1, i \leq \theta - 3; \\ y_i^2 &= \mu_i, \quad q_{ii} = -1; \quad y_{ij} = \lambda_{ij}, \quad i < j - 1, i \neq \theta - 2; \\ y_{(\theta-2\theta)} + q_{\theta-1\theta}(1 + q^{-1})[y_{\theta-2\theta}, y_{\theta-1}]_c - q_{\theta-2\theta-1}(1 - q^2)y_{\theta-1}y_{\theta-2\theta} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta}, y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of super type $A_{\theta-1}$ and another of type A_1 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_i^2 &= \mu_i, \quad q_{ii} = -1; \quad y_{ij} = \lambda_{ij}, \quad i < j - 1, i \neq \theta - 2; \\ y_{\theta-2\theta-2\theta} &= \lambda_{\theta-2\theta-2\theta}; \quad y_{iii\pm 1} = \lambda_{iii\pm 1}, \quad i < \theta - 1, q_{ii} \neq -1; \\ y_{\theta-1\theta}^2 &= \mu_{\theta-1\theta}; \quad [y_{(i-1i+1)}, y_i]_c = \nu_i, \quad q_{ii} = -1, i \leq \theta - 3; \\ y_{(\theta-2\theta)} + q_{\theta-1\theta}(1 + q^{-1})[y_{\theta-2\theta}, y_{\theta-1}]_c - q_{\theta-2\theta-1}(1 - q^2)y_{\theta-1}y_{\theta-2\theta} &= 0. \end{aligned}$$

Note that $\lambda_{\theta-2\theta-2\theta}\lambda_{\theta-2\theta-2\theta-1} = 0$, so we may assume $\lambda_{\theta-2\theta-2\theta} = 0$. If $\mu_{\theta-1\theta} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta}, y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of super type $A_{\theta-1}$ and another of type A_1 . If $\mu_{\theta-1\theta} \neq 0$, $\lambda_{\theta-2\theta-2\theta-1} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta}, y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of super type $A_{\theta-2}$ and another of Cartan type A_2 . If $\mu_{\theta-1\theta}\lambda_{\theta-2\theta-2\theta-1} \neq 0$, $\lambda_{\theta-2\theta-2\theta-3} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{\theta-3\theta-2} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of super type $A_{\theta-2}$ and a subalgebra of that in **CDrk4c.log**. If $\mu_{\theta-1\theta}\lambda_{\theta-2\theta-2\theta-1}\lambda_{\theta-2\theta-2\theta-3} \neq 0$, then $\nu_{\theta-3} = 0$ and $\tilde{\mathcal{E}}_q/\langle y_{\theta-4\theta-3} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of super type $A_{\theta-4}$ and another with matrix $\begin{pmatrix} -1 & q & -1 & -1 \\ -1 & q & -1 & \pm 1 \\ -1 & q & -1 & 1 \\ -1 & \pm q^{-1} & -1 & -1 \end{pmatrix}$; this algebra is not zero, see **CDrk4c.log**.

For the diagram

$$\begin{array}{ccc} & \overset{-1}{\circ} & \\ & \downarrow q^{-1} & \searrow q^2 \\ \mathbf{A}_{\theta-3}(q; i_1, \dots, i_j) & \xrightarrow[q]{-1} \overset{-1}{\circ} & \xrightarrow[q^{-1}]{-1} \overset{-1}{\circ} \end{array} \quad \text{there are 2 cases:}$$

$\diamond N \neq 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} [y_{\theta-3\theta-2\theta}, y_{\theta-2}]_c &= \nu'_{\theta-2}; \quad y_{iii\pm 1} = \lambda_{iii\pm 1}, \quad i < \theta - 2, q_{ii} \neq -1; \\ [y_{(i-1i+1)}, y_i]_c &= \nu_i, \quad q_{ii} = -1, i \leq \theta - 3; \\ y_i^2 &= \mu_i, \quad q_{ii} = -1; \quad y_{ij} = \lambda_{ij}, \quad i < j - 1, i \neq \theta - 2; \\ y_{(\theta-2\theta)} + q_{\theta-1\theta}(1 + q^{-1})[y_{\theta-2\theta}, y_{\theta-1}]_c - q_{\theta-2\theta-1}(1 - q^2)y_{\theta-1}y_{\theta-2\theta} &= 0. \end{aligned}$$

Here, $\nu'_{\theta-2}\nu'_{\theta-2} = 0$, we assume $\nu'_{\theta-2} = 0$. Thus $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta}, y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, of super type $A_{\theta-1}$ and of type A_1 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_{ij} &= \lambda_{ij}, \quad i < j-1, i \neq \theta-2; & y_{\theta-1\theta}^2 &= \mu_{\theta-1\theta} \\ y_{iii\pm 1} &= \lambda_{iii\pm 1}, \quad i < \theta-2, q_{ii} \neq -1; & [y_{\theta-3\theta-2\theta}, y_{\theta-2}]_c &= \nu'_{\theta-2}; \\ [y_{(i-1i+1)}, y_i]_c &= \nu_i, \quad q_{ii} = -1, i \leq \theta-3; & y_i^2 &= \nu_i, \quad q_{ii} = -1; \\ y_{(\theta-2\theta)} + q_{\theta-1\theta}(1+q^{-1})[y_{\theta-2\theta}, y_{\theta-1}]_c - 2q_{\theta-2\theta-1}y_{\theta-1}y_{\theta-2\theta} &= 0. \end{aligned}$$

Note that $\nu_{\theta-2}\lambda'_{\theta-2} = 0$, so we may assume $\nu'_{\theta-2} = 0$. If $\mu_{\theta-1\theta} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta}, y_{\theta-1\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type $A_{\theta-1}$ and another of type A_1 . If $\mu_{\theta-1\theta} \neq 0, \nu_{\theta-2} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta}, y_{\theta-2\theta-1} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type $A_{\theta-2}$ and another of Cartan type A_2 . If $\mu_{\theta-1\theta}\nu_{\theta-2} \neq 0$, then $q_{\theta-3\theta-3} = -1$ and $\nu_{\theta-3} = 0$; hence $\tilde{\mathcal{E}}_q/\langle y_{\theta-4\theta-3}, y_{\theta-2\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type $A_{\theta-4}$ and another of Cartan type C_4 .

For the diagram

$$\begin{array}{c} \overset{q}{\circ} \\ \left| \begin{array}{c} q^{-1} \end{array} \right. \\ \mathbf{A}_{\theta-3}(q; i_1, \dots, i_j) \xrightarrow[q^{-1}]{} \overset{q}{\circ} \xrightarrow[q^{-1}]{} \overset{q}{\circ} \end{array} \quad \text{the algebra } \tilde{\mathcal{E}}_q \text{ is gen-}$$

erated by $y_i, i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_i^2 &= \nu_i, \quad q_{ii} = -1 & y_{iii\pm 1} &= \lambda_{iii\pm 1}, \quad i < \theta-1, q_{ii} \neq -1; \\ y_{\theta-2\theta-2\theta} &= \lambda_{\theta-2\theta-2\theta}; & y_{\theta-1\theta} &= 0; \\ y_{\theta-1\theta-1\theta-2} &= \lambda_{\theta-1\theta-1\theta-2}; & y_{ij} &= \lambda_{ij}, \quad i < j-1, i \neq \theta-2; \\ y_{\theta\theta\theta-2} &= \lambda_{\theta\theta\theta-2}; & [y_{(i-1i+1)}, y_i]_c &= \nu_i, \quad q_{ii} = -1, i \leq \theta-3. \end{aligned}$$

If $N \neq 3$, then $\lambda_{\theta-2\theta-2\theta} = \lambda_{\theta\theta\theta-2} = \lambda_{\theta-1\theta-1\theta-2} = \lambda_{\theta-2\theta-2\theta-1} = 0$; if $N = 3$, then either $\lambda_{\theta-2\theta-2\theta} = \lambda_{\theta\theta\theta-2} = 0$, or else $\lambda_{\theta-1\theta-1\theta-2} = \lambda_{\theta-2\theta-2\theta-1} = 0$. Thus we may assume $\lambda_{\theta-2\theta-2\theta} = \lambda_{\theta\theta\theta-2} = 0$, and $\tilde{\mathcal{E}}_q/\langle y_{\theta-2\theta} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of super type $A_{\theta-1}$ and another of type A_1 .

For the diagram

$$\begin{array}{c} \overset{q}{\circ} \\ \left| \begin{array}{c} q^{-1} \end{array} \right. \\ \mathbf{A}_{\theta-3}(q; i_1, \dots, i_j) \xrightarrow[q]{} \overset{-1}{\circ} \xrightarrow[q^{-1}]{} \overset{q}{\circ} \end{array} \quad \text{the algebra } \tilde{\mathcal{E}}_q \text{ is gen-}$$

erated by $y_i, i \in \mathbb{I}_\theta$, with defining relations

$$\begin{aligned} y_i^2 &= \mu_i, \quad q_{ii} = -1; & y_{iii\pm 1} &= \lambda_{iii\pm 1}, \quad i < \theta-2, q_{ii} \neq -1; \\ [y_{\theta-3\theta-2\theta}, y_{\theta-2}]_c &= \nu'_{\theta-2}; & y_{\theta-1\theta} &= 0; \\ y_{\theta-1\theta-1\theta-2} &= \lambda_{\theta-1\theta-1\theta-2}; & y_{ij} &= \lambda_{ij}, \quad i < j-1, i \neq \theta-2; \\ y_{\theta\theta\theta-2} &= \lambda_{\theta\theta\theta-2}; & [y_{(i-1i+1)}, y_i]_c &= \nu_i, \quad q_{ii} = -1, i \leq \theta-2. \end{aligned}$$

As above, we may restrict to $\theta = 4$ and $N = 4$. As $\nu_2\nu'_2 = 0$, hence we may assume $\nu'_2 = 0$. Assume that $q_{11} = -1$, then $\nu_2 = 0$ and we can quotient onto a case of smaller rank, by making $y_{12} = 0$. Otherwise, $q_{11} = q^{-1}$. If only $\nu_2 \neq 0$, we can project onto smaller ranks. Hence, we can restrict to the case $\nu_2 = 0$ as described in §5.1.5. We can further project to smaller rank

unless $\lambda_{112}\lambda_{332}\lambda_{442} \neq 0$, when $\mathfrak{q} = \begin{pmatrix} q^{-1} & -1 & a^{-1}q & b^{-1} \\ q^{-1} & -1 & q & q \\ a & -1 & q & c^{-1} \\ b & -1 & c & q \end{pmatrix}$, for $a, b, c \in \{\pm q\}$. In

any case, $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$, see `CDrk4d.log`.

8.4. Type $\mathbf{D}(2, 1; \alpha)$, $q, r, s \neq 1$, $qrs = 1$. The Weyl groupoid has four objects.

For $\begin{array}{ccccc} & & q & & \\ & \nearrow & & \searrow & \\ -1 & \xrightarrow{s} & -1 & \xrightarrow{r} & -1 \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^2 &= \lambda_3; \\ y_{(13)} &= \frac{1-s}{q_{23}(1-r)}[y_{13}, y_2]_c - q_{12}(1-s)y_2y_{13} = 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{ccccc} q & \xrightarrow{q^{-1}} & -1 & \xrightarrow{r^{-1}} & r \\ \circ & & \circ & & \circ \end{array}$, $q, r, s \neq -1$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_2^2 = \lambda_2; \quad y_{112} = \lambda_1; \quad y_{332} = \lambda_3; \quad y_{13} = 0,$$

with $\lambda_1\lambda_3 = 0$. This algebra is nonzero, as it projects onto a nonzero algebra by making $y_1 = 0$ or $y_3 = 0$ according to $\lambda_1 = 0$ or $\lambda_3 = 0$.

For the same diagram and $q = -1, r, s \neq -1$, $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_1^2 = \lambda_1; \quad y_2^2 = \lambda_2; \quad y_{12}^2 = \lambda_4; \quad y_{332} = \lambda_3; \quad y_{13} = 0.$$

Here $\lambda_3 \neq 0$ only if $r \in G_4$. If $\lambda_3 = 0$, then this algebra is nonzero as it projects onto an algebra of Cartan type A_2 . Similarly, a projection to the rank-two case shows that $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ if $\lambda_1 = \lambda_2 = \lambda_4 = 0$.

Assume $\lambda_3 \neq 0$ and pick $\epsilon \in \mathbb{k}$ such that $\epsilon^2 = 1$, i.e. $\epsilon = \pm r^2$. Then: $\lambda_1\lambda_4 \neq 0$ only if $\mathfrak{q} = \begin{pmatrix} -1 & 1 & \epsilon \\ -1 & -1 & r \\ \epsilon & r^2 & r \end{pmatrix}$. $\lambda_2 \neq 0$ only if $\mathfrak{q} = \begin{pmatrix} -1 & \epsilon & x \\ -\epsilon & -1 & r \\ x^{-1} & r^2 & r \end{pmatrix}$, with $x^2 = \epsilon$.

In any case, $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$, see `d21a.log`.

The remaining two generalized Dynkin diagrams are of the shape of the last one, but with s instead of q , respectively with s instead of r .

8.5. **Type F(4), $N > 3$.** The Weyl groupoid has objects with generalized Dynkin diagrams of six possible shapes.

For the diagram $\overset{q^2}{\circ} \xrightarrow{q^{-2}} \overset{q^2}{\circ} \xrightarrow{q^{-2}} \overset{q}{\circ} \xrightarrow{q^{-1}} \overset{-1}{\circ}$, there are 2 cases:

$\diamond N > 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{13} &= 0; & y_{14} &= 0; & y_{24} &= 0; \\ y_{112} &= 0; & y_{221} &= 0; & y_{223} &= \lambda_{223}; \\ y_{334} &= 0; & y_{3332} &= \lambda_{3332}; & y_4^2 &= \mu_4. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type B_3 and another of type A_1 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{13} &= 0; & y_{14} &= \lambda_{14}; & y_{24} &= \lambda_{24}; \\ y_{334} &= \lambda_{334}; & [y_{(13)}, y_2]_c &= 0; & [y_{23}, y_{(24)}]_c &= \lambda_{(24)}; \\ y_{3332} &= 0; & y_4^2 &= \mu_4; & y_{12}^2 &= \mu_{12}. \end{aligned}$$

If $\lambda_{(24)} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type A_2 and another of super type A_2 . If $\lambda_{(24)} \neq 0$ but all the other λ 's are 0, then $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of type super C_3 and A_1 . The general case follows by applying Lemma 3.3.

For the diagram $\overset{q^2}{\circ} \xrightarrow{q^{-2}} \overset{q^2}{\circ} \xrightarrow{q^{-2}} \overset{-1}{\circ} \xrightarrow{q} \overset{-1}{\circ}$, there are 2 cases:

$\diamond N > 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{13} &= 0; & y_{112} &= \lambda_{112}; & y_{24} &= 0; & y_3^2 &= \mu_3; \\ y_{14} &= 0; & y_{221} &= \lambda_{221}; & y_{223} &= 0; & y_4^2 &= \mu_4; & [[y_{43}, y_{432}]_c, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type A_2 and another of super type A_2 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{13} &= \lambda_{13}; & y_{14} &= \lambda_{14}; & [y_{(13)}, y_2]_c &= \nu_2; & y_3^2 &= \mu_3; \\ y_{23}^2 &= \mu_{23}; & y_{24} &= \lambda_{24}; & [[y_{43}, y_{432}]_c, y_3]_c &= \lambda_{432}; & y_4^2 &= \mu_4. \end{aligned}$$

If $\nu_2 = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type A_1 and another of super type CD . If $\nu_2 \neq 0$, then $\lambda_{432} = 0$ and $\tilde{\mathcal{E}}_q/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, of types A_3 and A_1 .

For the diagram $\overset{q^2}{\circ} \xrightarrow{q^{-2}} \overset{q^2}{\circ} \xrightarrow{q^{-2}} \overset{-1}{\circ} \xrightarrow{q^3} \overset{q^{-3}}{\circ}$, there are 3 cases:

$\diamond N \neq 4, 6$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{443} &= \lambda_{443}; & y_3^2 &= \mu_3; & y_{13} &= \lambda_{13}; & y_{14} &= 0; & y_{24} &= 0; \\ y_{112} &= 0; & y_{221} &= 0; & y_{223} &= \lambda_{223}; & [[y_{432}, y_3]_c, [y_{4321}, y_3]_c]_c, y_{32}]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of type A_1 and another of super type $D(2, 1; \alpha)$.

$\diamond N = 6$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_2; & y_{34}^2 &= 0; & y_{13} &= 0; & y_{14} &= 0; & y_{24} &= 0; \\ y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= 0; & [[y_{432}, y_3]_c, [y_{4321}, y_3]_c]_c, y_{32} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of type A_2 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & y_{13} &= \lambda_{13}; & y_{14} &= 0; & [y_{(13)}, y_2]_c &= \nu_2; \\ y_{443} &= \lambda_{443}; & y_{23}^2 &= \mu_{23}; & y_{24} &= 0; & [[y_{432}, y_3]_c, [y_{4321}, y_3]_c]_c, y_{32} &= 0. \end{aligned}$$

If $\nu_2 = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of type A_1 and another of super type CD . If $\nu_2 \neq 0$, then $\lambda_{443} = 0$ and $\tilde{\mathcal{E}}_q/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, of types A_3 and A_1 .

For the diagram $\begin{array}{ccccccc} & q^2 & q^{-2} & q & q^{-1} & -1 & q^3 & q^{-3} \\ & \circ & \xrightarrow{\quad} & \circ & \xrightarrow{\quad} & \circ & \xrightarrow{\quad} & \circ \end{array}$, there are 3 cases:

$\diamond N \neq 4, 6$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & y_{13} &= 0; & y_{14} &= 0; & y_{24} &= 0; \\ y_{443} &= \lambda_{443}; & y_{112} &= \lambda_{221}; & y_{2221} &= \lambda_{2221}; & y_{223} &= 0; \\ & & [[y_{(14)}, y_2]_c, y_3]_c - q_{23}(q^2 - q)[[y_{(14)}, y_3]_c, y_2]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of type B_2 and another of super type A_2 .

$\diamond N = 6$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & y_{13} &= 0; & y_{14} &= 0; & y_{24} &= 0; \\ y_{34}^2 &= \mu_{34}; & y_{112} &= 0; & y_{2221} &= 0; & y_{223} &= 0; \\ & & [[y_{(14)}, y_2]_c, y_3]_c - q_{23}(q^2 - q)[[y_{(14)}, y_3]_c, y_2]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of type B_2 and another of super type A_2 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & y_{13} &= \lambda_{13}; & y_{14} &= 0; & y_{24} &= 0; \\ y_{443} &= \lambda_{443}; & [y_{12}, y_{(13)}]_c &= \lambda_{(13)}; & y_{2221} &= 0; & y_{223} &= \lambda_{223}; \\ & & [[y_{(14)}, y_2]_c, y_3]_c - q_{23}(q^2 - q)[[y_{(14)}, y_3]_c, y_2]_c &= 0. \end{aligned}$$

If $\lambda_{(13)} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, one of type A_1 and another of type $D(2, 1; \alpha)$. If $\lambda_{(13)} \neq 0$ but all the other λ 's are 0, then $\tilde{\mathcal{E}}_q/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types B_3 and A_1 . The general case follows by applying Lemma 3.3.

For the diagram

$$\begin{array}{ccccc} & & q & & \\ & & \circ & & \\ & q^{-1} \swarrow & & \downarrow q^{-1} & \\ q^2 & \xrightarrow{q^{-2}} & -1 & \xrightarrow{q^2} & -1 \\ & \circ & & \circ & \end{array}$$

there are 2 cases:

$\diamond N > 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_2^2 &= \mu_2; & y_{13} &= 0; & y_{14} &= 0; & y_{112} &= \lambda_{112}; \\ y_3^2 &= \mu_3; & y_{442} &= 0; & y_{443} &= 0; & [y_{(13)}, y_2]_c &= 0; \\ y_{(24)} - q_{34}q[y_{24}, y_3]_c - q_{23}(1 - q^{-1})y_3y_{24} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{23}, y_{24} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of super type A_2 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_2^2 &= \mu_2; & y_{13} &= \lambda_{13}; & y_{14} &= 0; & y_{12}^2 &= \mu_{12}; \\ y_3^2 &= \mu_3; & y_{442} &= \lambda_{442}; & y_{443} &= \lambda_{443}; & [y_{(13)}, y_2]_c &= \nu_2; \\ y_{(24)} - q_{34}q[y_{24}, y_3]_c - q_{23}(1 - q^{-1})y_3y_{24} &= 0. \end{aligned}$$

If $\lambda_{442} \neq 0$, then $\nu_2 = \lambda_{443} = 0$, so $\tilde{\mathcal{E}}_q/\langle y_{23}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, of types A_1 and C_3 . If $\lambda_{443} \neq 0$, then $\nu_2 = \lambda_{442} = 0$, so $\tilde{\mathcal{E}}_q/\langle y_{23}, y_{24} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, of types A_2 and super A_2 . If $\nu_2 \neq 0$, then $\lambda_{442} = \lambda_{443} = 0$, so $\tilde{\mathcal{E}}_q/\langle y_{24}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components, of types A_1 and A_3 .

For the diagram

$$\begin{array}{ccccc} & & -1 & & \\ & & \circ & & \\ & q^2 \swarrow & & \downarrow q^{-3} & \\ q^2 & \xrightarrow{q^{-2}} & -1 & \xrightarrow{q} & -1 \\ & \circ & & \circ & \end{array}$$

there are 2 cases:

$\diamond N > 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_2^2 &= \mu_2; & y_{13} &= 0; & y_{14} &= 0; & [y_{124}, y_2]_c &= 0; \\ y_3^2 &= \mu_3; & y_{112} &= \lambda_{112}; & y_4^2 &= \mu_4; & [[y_{32}, y_{321}]_c, y_2]_c &= 0; \\ y_{(24)} + q_{34} \frac{1 - q^3}{1 - q^2} [y_{24}, y_3]_c - q_{23}(1 - q^{-3})y_3y_{24} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{23}, y_{24} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of super type A_2 .

$\diamond N = 4$. Then $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_2^2 &= \mu_2; & y_{13} &= \lambda_{13}; & y_{14} &= \lambda_{14}; & [y_{124}, y_2]_c &= \nu_2; \\ y_3^2 &= \mu_3; & y_{12}^2 &= \mu_{12}; & y_4^2 &= \mu_4; & [[y_{32}, y_{321}]_c, y_2]_c &= \lambda_{(321)}; \\ y_{(24)} + q_{34} \frac{1 - q^3}{1 - q^2} [y_{24}, y_3]_c - q_{23}(1 - q^{-3})y_3y_{24} &= 0. \end{aligned}$$

If $\lambda_{(321)} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{23}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components, of types A_1 and A_3 . If $\lambda_{(321)} \neq 0$, then $\nu_2 = 0$, so $\tilde{\mathcal{E}}_q/\langle y_{24}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components, of types A_1 and A_3 .

8.6. Type $\mathbf{G}(3)$, $N > 3$. The Weyl groupoid has four objects.

For $\begin{smallmatrix} -1 & q^{-1} & q & q^{-3} & q^3 \\ \circ & & \circ & & \circ \end{smallmatrix}$, $N \neq 4, 6$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = 0; \quad y_{221} = 0; \quad y_{332} = 0; \quad y_{22223} = \lambda_2; \quad y_1^2 = \lambda_1,$$

with $\lambda_1 \lambda_2 = 0$. If $\lambda_1 \neq 0$, then $\tilde{\mathcal{E}}_q$ projects onto $\mathbb{k}\langle y_1 | y_1^2 = \lambda_1 \rangle$. If $\lambda_2 \neq 0$, then $\tilde{\mathcal{E}}_q$ projects onto a nonzero algebra related to Cartan case G_2 .

When $N = 6$, $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = \lambda_3; \quad y_{221} = 0; \quad [y_{12}, y_{(13)}]_c = 0; \quad y_{22223} = 0; \quad y_1^2 = \lambda_1.$$

Now $\tilde{\mathcal{E}}_q \neq 0$ by projecting onto the case $y_2 = 0$.

For $N = 4$, $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = 0; \quad y_{221} = \lambda_2; \quad [[[y_{(13)}, y_2]_c, y_2]_c, y_2]_c = 0; \quad y_{332} = 0; \quad y_1^2 = \lambda_1.$$

The longest relation is undeformed by **g3aN4.log**. Here $\tilde{\mathcal{E}}_q$ projects onto a nonzero algebra related to Cartan case A_2 .

For $\begin{smallmatrix} -1 & q & -1 & q^{-3} & q^3 \\ \circ & & \circ & & \circ \end{smallmatrix}$, $N \neq 6$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$,

with defining relations

$$y_{13} = 0; \quad y_{332} = \lambda_3; \quad [[y_{12}, [y_{12}, y_{(13)}]_c]_c, y_2]_c = 0; \quad y_1^2 = \lambda_1; \quad y_2^2 = \lambda_2.$$

The third relation is undeformed by **g3b.log**. Here $\lambda_3 \neq 0$ only when $N = 12$. In any case $\tilde{\mathcal{E}}_q \neq 0$ by making $y_{12} = 0$ as in §5.1.3.

When $N = 6$, $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = \lambda_3; \quad y_{23}^2 = \lambda_5; \quad [[y_{12}, [y_{12}, y_{(13)}]_c]_c, y_2]_c = \lambda_4; \quad y_1^2 = \lambda_1; \quad y_2^2 = \lambda_2.$$

If $\lambda_4 = 0$, then we can project onto a case of smaller rank again making

$y_{12} = 0$. If $\lambda_4 \neq 0$, then $\mathfrak{q} = \begin{pmatrix} -1 & a & -a^{-4} \\ a^{-1}q & -1 & -a^3 \\ -a^4 & a^{-3} & -1 \end{pmatrix}$. Here, $\tilde{\mathcal{E}}_q \neq 0$ by **g3b2.log**.

For $\begin{smallmatrix} -q^{-1} & q^2 & -1 & q^{-3} & q^3 \\ \circ & & \circ & & \circ \end{smallmatrix}$, $N \neq 6$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_3$, with defining relations

$$y_{13} = \lambda_3; \quad y_{332} = \lambda_4; \quad y_{1112} = 0; \quad y_2^2 = \lambda_2;$$

$$[y_1, [y_{123}, y_2]_c]_c = \frac{q_{12}q_{32}}{1+q} [y_{12}, y_{123}]_c - q^{-1}(1 - q^{-1})q_{12}q_{13}y_{123}y_{12}.$$

Here $\lambda_3 \neq 0$ only if $N = 4$ and $\lambda_4 \neq 0$ only if $N = 12$; also $\lambda_2 \lambda_3 = 0$. Hence, we can project onto a case of smaller rank and $\tilde{\mathcal{E}}_q \neq 0$. The longest relation is undeformed by **g3c.log**.

When $N = 6$, $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{13} &= 0; & y_{23}^2 &= \lambda_3; & [y_{112}, y_{12}]_c &= \lambda_1; & y_2^2 &= \lambda_2; \\ [y_1, [y_{123}, y_2]_c]_c &= \frac{q_{12}q_{32}}{1+q} [y_{12}, y_{123}]_c - q^{-1}(1-q^{-1})q_{12}q_{13}y_{123}y_{12}. \end{aligned}$$

We see that $\tilde{\mathcal{E}}_q \neq 0$ using the arguments in §5.1.5. The longest relation is undeformed by `g3c.log`.

For $\begin{array}{c} q^{-1} \\ \text{---} \text{---} \text{---} \\ \circ \xrightarrow{q^{-2}} \circ \xrightarrow{-1} \circ \xrightarrow{q^3} \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{1112} &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^2 &= \lambda_3; & y_{113} &= \lambda_4; \\ y_{(13)} + q^{-2}q_{23} \frac{1-q^3}{1-q} [y_{13}, y_2]_c - q_{12}(1-q^3)y_2y_{13} &= 0. \end{aligned}$$

If $\lambda_1 = \lambda_4 = 0$, then $\tilde{\mathcal{E}}_q \neq 0$ by [A+, Lemma 5.16]. Now, $\lambda_1 \neq 0$ only if $N = 6$ and $\lambda_4 \neq 0$ only if $N = 4$. If $\lambda_1 \neq 0$ and $\lambda_3 = 0$ or if $\lambda_4 \neq 0$ and $\lambda_2 = 0$ we may project onto a case of smaller rank. Indeed, $\lambda_2\lambda_4 \neq 0$ and we can have $\lambda_1\lambda_3 \neq 0$ only if $\mathfrak{q} = \begin{pmatrix} q & -1 & \pm 1 \\ q & -1 & \pm 1 \\ \pm q^{-1} & \pm q^{-1} & -1 \end{pmatrix}$. In this case, $\tilde{\mathcal{E}}_q \neq 0$ by `g3d.log`.

9. MODULAR TYPE, CHARACTERISTIC 3

9.1. **Type** `br(2, a)`, $\zeta \in G_3$, $q \notin G_3$. The Weyl groupoid has two objects:

$$\begin{array}{c} \zeta \\ \text{---} \text{---} \text{---} \\ \circ \xrightarrow{q^{-1}} \circ \end{array} \quad \text{and} \quad \begin{array}{c} \zeta \\ \text{---} \text{---} \text{---} \\ \circ \xrightarrow{\zeta^2 q} \circ \end{array}.$$

Notice that the right hand diagram is of the shape of the left hand one, with ζq^{-1} instead of q . Therefore, we only have to analyze one of them: we concentrate on the later.

If $q \neq -1$, then the algebra $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_{221} = \lambda_2,$$

with $\lambda_1\lambda_2 = 0$. Also, $\lambda_2 = 0$ unless $q = -\zeta$ and $\mathfrak{q} = \begin{pmatrix} \zeta & -\zeta \\ \zeta & -\zeta \end{pmatrix}$. This algebra is nonzero: If $\lambda_1 \neq 0$, this is [A+, Lemma 5.16]. When $\lambda_2 \neq 0$, see `br2a-q.log`.

If $q = -1$, then the algebra $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [y_{112}, y_{12}]_c = \lambda_3,$$

with, $\lambda_3 = 0$ unless $\mathfrak{q} = \begin{pmatrix} \zeta & -1 \\ 1 & -1 \end{pmatrix}$. This algebra is nonzero, see `br2a-1.log`.

9.2. **Type** $\mathfrak{br}(3)$, $\zeta \in G'_9$. The Weyl groupoid has two objects.

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \zeta^3$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = 0; \quad y_{112} = 0; \quad y_3^3 = \lambda_1; \quad [[y_{332}, y_{3321}]_c, y_{32}]_c = 0; \quad y_{221} = 0; \quad y_{223} = 0.$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\zeta \xrightarrow{\bar{\zeta}} \zeta^4 \xrightarrow{\bar{\zeta}^3}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = 0; \quad y_{112} = 0; \quad y_3^3 = \lambda_1; \quad y_{2221} = 0; \quad y_{223} = 0; \\ [[y_{(13)}, y_2]_c, y_3]_c - (1 + \zeta^4)^{-1} q_{23} [[y_{(13)}, y_3]_c, y_2]_c = 0.$$

This algebra is nonzero by [A+, Lemma 5.16].

10. SUPER MODULAR TYPE, CHARACTERISTIC 3

10.1. **Type** $\mathfrak{brj}(2; 3)$, $\zeta \in G'_9$. The Weyl groupoid has three objects.

For $-\zeta \xrightarrow{\bar{\zeta}^2} \zeta^3$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_2^3 = \lambda_2; \quad [y_1, [y_{12}, y_2]_c]_c + \frac{\zeta^5(1 - \zeta)q_{12}}{1 - \zeta^7} y_{12}^2 = 0; \quad y_{1112} = 0.$$

This algebra is nonzero as it projects onto $\mathbb{k}\langle y_2 | y_2^3 = \lambda_2 \rangle$.

For $\zeta^3 \xrightarrow{\bar{\zeta}} -1$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [y_{112}, [y_{112}, y_{12}]_c]_c = 0,$$

with $\lambda_1 \lambda_2 = 0$. The last relation is undeformed, see `brj23.log`. Hence this algebra is nonzero by [A+, Lemma 5.16].

For $-\zeta^2 \xrightarrow{\zeta} -1$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_2^2 = \lambda_2; \quad [y_{112}, y_{12}]_c = 0; \quad y_{111112} = 0.$$

This algebra is nonzero by [A+, Lemma 5.16].

10.2. **Type** $\mathfrak{g}(1, 6)$, $\zeta \in G'_3 \cup G'_6$. The Weyl groupoid has two objects.

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}^2} -1$, $\zeta \in G'_6$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = 0; \quad y_{112} = 0; \quad y_{221} = 0; \quad y_{2223} = \lambda_{2223}; \quad y_3^2 = \mu_3.$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of super type A_2 and A_1 .

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{-\bar{\zeta}} \zeta^2 \xrightarrow{-1}$, $\zeta \in G'_6$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_{112} =; \quad y_3^2 = \mu_3; \quad [y_{223}, y_{23}]_c = \lambda_{23}; \\ [y_2, [y_{21}, y_{23}]_c]_c + q_{13}q_{23}q_{21}[y_{223}, y_{21}]_c + q_{21}y_{21}y_{223} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of super type B_2 and A_1 .

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}^2} \zeta^2 \xrightarrow{-1}$, $\zeta \in G'_3$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_{112} = \lambda_{112}; \quad [[y_{(13)}, y_2]_c, y_2]_c = 0; \\ y_3^2 = \mu_3; \quad y_{221} = \lambda_{221}; \quad [y_{223}, y_{23}]_c = \lambda_{23}. \end{aligned}$$

If $\lambda_{23} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, of types A_2 and A_1 . If $\lambda_{23} \neq 0$, then $\lambda_{112} = \lambda_{221} = 0$, so $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, of types A_1 and super B_2 .

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{-\bar{\zeta}} \zeta^2 \xrightarrow{-1}$, $\zeta \in G'_3$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_{112} = 0; \quad y_{2221} = 0; \quad y_{2223} = \lambda_{2223}; \quad y_3^2 = \mu_3; \\ [y_1, y_{223}]_c + q_{23}[y_{(13)}, y_2]_c + (\zeta^2 - \zeta)q_{12}y_2y_{(13)} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of super type B_2 and A_1 .

10.3. Type $\mathfrak{g}(2, 3)$, $\zeta \in G'_3$. The Weyl groupoid has four objects.

For $\zeta \xrightarrow{-1} \zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{-1}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{13} = \lambda_{13}; \quad y_{221} = 0; \quad y_1^2 = \mu_1; \quad y_3^2 = \mu_3; \\ [y_{223}, y_{23}]_c = \lambda_{23}; \quad [[y_{(13)}, y_2]_c, y_2]_c = \lambda_{(13)}. \end{aligned}$$

If $\lambda_{(13)} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type A_1 and another of type super B_2 . If $\lambda_{(13)} \neq 0$, then $q = \begin{pmatrix} -1 & b & -b^{-3} \\ \zeta^2 b^{-1} & \zeta & \zeta b \\ -b^3 & b^{-1} & -1 \end{pmatrix}$, $b \neq 0$; this algebra is not zero, see `g23-1.log`.

For $\zeta \xrightarrow{-1} \zeta \xrightarrow{-1} \zeta \xrightarrow{-1}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{13} = \lambda_{13}; \quad y_1^2 = \mu_1; \quad y_2^2 = \mu_2; \quad y_3^2 = \mu_3; \\ [[y_{12}, y_{(13)}]_c, y_2]_c = \lambda_{123}; \quad [[y_{32}, y_{321}]_c, y_2]_c = \lambda_{321}. \end{aligned}$$

If $\lambda_{123} = \lambda_{321} = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type A_1 and another of type super A_2 . If $\lambda_{123} \neq 0$ (similar for $\lambda_{321} \neq 0$),

then $\mathfrak{q} = \begin{pmatrix} -1 & b & b^{-3} \\ \zeta b^{-1} & -1 & -\zeta b^2 \\ b^3 & -b^{-2} & -1 \end{pmatrix}$, $b \neq 0$; this algebra is not zero, see `g23-2.log`.

Observe that $\lambda_{123}\lambda_{321} \neq 0$ only if $b^3 = -1$.

For $\overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{-\bar{\zeta}}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{13} &= \lambda_{13}; & y_{2221} &= \lambda_{2221}; & y_{2223} &= \lambda_{2223}; & y_1^2 &= \mu_1; & y_3^2 &= \mu_3; \\ [y_1, y_{223}]_c + q_{23}[y_{(13)}, y_2]_c - (1 - \zeta)q_{12}y_2y_{(13)} &= 0. \end{aligned}$$

As $\lambda_{2221}\lambda_{2223} = 0$, we may assume $\lambda_{2221} = 0$, and $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of type super B_2 , and other of type A_1 .

For $\overset{\bar{\zeta}}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{\zeta}{\circ}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{113} &= 0; & y_{331} &= 0; & y_{332} &= 0; & y_2^2 &= \mu_2; \\ y_{(13)} - q_{23}\zeta[y_{13}, y_2]_c - q_{12}(1 - \bar{\zeta})y_2y_{13} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{12}, y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, of type A_2 and A_1 .

10.4. **Type $\mathfrak{g}(3, 3)$** , $\zeta \in G'_3$. The Weyl groupoid has six objects.

For $\overset{\bar{\zeta}}{\circ} \xrightarrow{\zeta} \overset{\bar{\zeta}}{\circ} \xrightarrow{\zeta} \overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ}$ the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{223} &= 0; & y_{13} &= \lambda_{13}; & y_{14} &= 0; & y_{112} &= \lambda_{112}; & [y_{3321}, y_{32}]_c &= \lambda_{321}; \\ y_{334} &= 0; & y_4^2 &= \mu_4; & y_{24} &= 0; & y_{221} &= \lambda_{221}; & [[y_{(24)}, y_3]_c, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, of types B_3 and A_1 .

For $\overset{\bar{\zeta}}{\circ} \xrightarrow{\zeta} \overset{\bar{\zeta}}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$ the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{13} &= 0; & y_3^2 &= \mu_3; & y_{14} &= 0; & y_{112} &= \lambda_{112}; & [[y_{43}, y_{432}]_c, y_3]_c &= 0; \\ y_4^2 &= 0; & y_{24} &= 0; & y_{221} &= \lambda_{221}; & y_{223} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, of types super A_3 and A_1 .

For $\overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$ the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$y_{13} = 0; \quad y_3^2 = \mu_3; \quad y_{14} = 0; \quad y_{34} = \lambda_{34}; \quad [y_{124}, y_2]_c = 0;$$

$$y_4^2 = \mu_4; \quad y_{112} = 0; \quad y_2^2 = \mu_2; \quad [y_{324}, y_2]_c = \nu_2.$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components, one of type super A_3 and another of type A_1 .

For $\begin{array}{c} \circ^{-1} \\ | \zeta \\ \bar{\zeta} \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } \circ^{-1} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = 0; \quad y_3^2 = \mu_3; \quad y_{14} = 0; \quad y_{34} = \lambda_{34}; \quad y_{112} = \lambda_{112}; \\ y_4^2 = \mu_4; \quad y_{221} = \lambda_{221}; \quad y_{223} = 0; \quad y_{224} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components, one of type super A_3 and another of type A_1 .

For $\begin{array}{c} \circ^{-1} \\ | \bar{\zeta} \\ \bar{\zeta} \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } \circ^{-1} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = 0; \quad y_3^2 = \mu_3; \quad y_{14} = 0; \quad y_{34} = \lambda_{34}; \quad [[y_{(13)}, y_2]_c, y_2]_c = 0; \\ y_{112} = 0; \quad y_4^2 = \mu_4; \quad y_{223} = 0; \quad y_{224} = 0; \quad [[y_{124}, y_2]_c, y_2]_c = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components, one of type super A_3 and another of type A_1 .

For $\begin{array}{c} \circ^{-1} \\ | \quad \searrow \bar{\zeta} \\ \bar{\zeta} \quad \bar{\zeta} \\ \bar{\zeta} \text{ --- } \zeta \text{ --- } \circ^{-1} \text{ --- } \bar{\zeta} \text{ --- } \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = \lambda_{13}; \quad y_2^2 = \mu_2; \quad y_{14} = 0; \quad y_{112} = 0; \quad [y_{(13)}, y_2]_c = \nu_2; \\ y_4^2 = \mu_4; \quad y_{332} = 0; \quad y_{334} = 0; \quad [y_{124}, y_2]_c = 0; \\ y_{(24)} - \zeta q_{34}[y_{24}, y_3]_c - q_{23}(1 - \bar{\zeta})y_3y_{24} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{24}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components, one of type super A_3 and another of type A_1 .

10.5. **Type $\mathfrak{g}(4, 3)$, $\zeta \in G'_3$.** The Weyl groupoid has ten objects.

For $\begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_{13} &= \lambda_{13}; & y_2^2 &= \mu_2; & y_{14} &= 0; & y_{24} &= \lambda_{14}; & [[y_{(24)}, y_3]_c, y_3]_c &= \lambda_{(24)}; \\ [y_{(13)}, y_2]_c &= \nu_2; & y_4^2 &= \mu_4; & y_{112} &= 0; & y_{332} &= 0; & [y_{334}, y_{34}]_c &= \lambda_{34}. \end{aligned}$$

If $\nu_2 = 0$, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 . If $\nu_2 \neq 0$, then $\lambda_{(24)} = \lambda_{34} = 0$, so $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super A_3 and another of type A_1 .

For $\begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \bar{\zeta} \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_{13} &= 0; & y_4^2 &= \mu_4; & y_{14} &= 0; & y_{24} &= \lambda_{24}; & [y_{(13)}, y_2]_c &= 0; \\ y_2^2 &= \mu_2; & y_{112} &= 0; & y_{3332} &= \lambda_{3332}; & y_{3334} &= \lambda_{3334}; \\ [y_2, y_{334}]_c - q_{34}[y_{(24)}, y_3]_c + (\zeta^2 - \zeta)q_{23}y_3y_{(24)} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 .

For $\begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & y_2^2 &= \mu_2; & y_{13} &= \lambda_{13}; & [y_{(13)}, y_2]_c &= \nu_2; \\ y_3^2 &= \mu_3; & y_{14} &= \lambda_{14}; & [[y_{23}, y_{(24)}]_c, y_3]_c &= \lambda_{234}; \\ y_4^2 &= \mu_4; & y_{24} &= \lambda_{24}; & [[y_{43}, y_{432}]_c, y_3]_c &= \lambda_{432}. \end{aligned}$$

If $\nu_2 = 0$, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 . If $\nu_2 \neq 0$, then $\lambda_{234} = \lambda_{432} = 0$, so $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super A_3 and another of type A_1 .

For $\begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_{13} &= \lambda_{13}; & y_3^2 &= \mu_3; & y_{14} &= \lambda_{14}; & y_{24} &= 0; & [[y_{43}, y_{432}]_c, y_3]_c &= 0; \\ y_4^2 &= \mu_4; & y_{221} &= 0; & y_{223} &= 0; & y_1^2 &= \mu_2. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} [y_{(13)}, y_2]_c &= 0; & y_1^2 &= \mu_1; & y_{13} &= 0; & [[y_{(24)}, y_3]_c, y_3]_c &= \lambda_{(24)}; \\ y_{334} &= 0; & y_2^2 &= \mu_2; & y_{14} &= \lambda_{14}; & [y_{3321}, y_{32}]_c &= 0; \end{aligned}$$

$$y_4^2 = \mu_4; \quad y_{24} = \lambda_{24}; \quad [y_{332}, y_{32}]_c = \lambda_{32}.$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 .

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_1^2 = \mu_1; \quad y_{14} = \lambda_{14}; \quad y_{24} = 0; \quad [[y_{(24)}, y_3]_c, y_3]_c = 0; \\ y_{334} = 0; \quad y_4^2 = \mu_4; \quad y_{221} = 0; \quad y_{223} = 0; \quad [y_{3321}, y_{32}]_c = 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \bar{\zeta} \\ \circ \\ \downarrow \zeta \\ \zeta \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} -1 \\ \circ \end{array} \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = 0; \quad y_2^2 = \mu_2; \quad y_{14} = \lambda_{14}; \quad y_{34} = 0; \quad [y_{124}, y_2]_c = \nu'_2; \\ [y_{(13)}, y_2]_c = 0; \quad y_3^2 = \mu_3; \quad y_{112} = 0; \quad y_{442} = 0; \quad [[y_{32}, y_{324}]_c, y_2]_c = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type super CD and another of type A_1 .

For $\begin{array}{c} -1 \\ \circ \\ \downarrow \bar{\zeta} \\ \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \zeta \\ \circ \end{array} \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = 0; \quad y_{14} = \lambda_{14}; \quad y_{221} = 0; \quad y_{223} = \lambda_{223}; \quad y_{224} = 0; \\ y_{332} = \lambda_{332}; \quad y_{334} = 0; \quad y_1^2 = \mu_1; \quad y_4^2 = \mu_4; \\ y_{(24)} - \zeta q_{34}[y_{24}, y_3]_c - (1 - \bar{\zeta})q_{23}y_3y_{24} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{24}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components, one of type super A_3 and another of type A_1 .

For $\begin{array}{c} \bar{\zeta} \\ \circ \\ \downarrow \zeta \\ \zeta \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = 0; \quad y_{442} = 0; \quad y_{14} = \lambda_{14}; \quad y_{34} = 0; \quad [[y_{124}, y_2]_c, y_2]_c = \lambda_{124}; \\ y_3^2 = 0; \quad y_{223} = 0; \quad y_{112} = \lambda_{112}; \quad y_{221} = \lambda_{221}; \quad [[y_{324}, y_2]_c, y_2]_c = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of type C_3 and another of type A_1 .

For $\begin{array}{c} \circ \\ \begin{array}{c} \overline{\zeta} \\ \downarrow \end{array} \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \searrow \\ \zeta \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = 0; \quad y_2^2 = \mu_2; \quad y_{14} = \lambda_{14}; \quad y_{332} = 0; \quad y_{334} = 0; \\ y_4^2 = \mu_4; \quad [y_{(13)}, y_2]_c = 0; \quad y_1^2 = \mu_1; \quad [y_{124}, y_2]_c = \nu'_2; \\ y_{(24)} - \zeta q_{34}[y_{24}, y_3]_c - (1 - \overline{\zeta})q_{23}y_3y_{24} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{23}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of type super A_3 and another of type A_1 .

10.6. **Type $\mathfrak{g}(3, 6)$** , $\zeta \in G'_3$. The Weyl groupoid has seven objects.

For $\begin{array}{c} \circ \\ \begin{array}{c} \overline{\zeta} \\ \downarrow \end{array} \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_4^2 = \mu_4; \quad y_{14} = \lambda_{14}; \quad y_{24} = 0; \quad [[y_{(24)}, y_3]_c, y_3]_c = 0; \\ y_{332} = \lambda_{332}; \quad y_1^2 = 0; \quad y_{221} = 0; \quad y_{223} = \lambda_{223}; \quad [y_{334}, y_{34}]_c = \lambda_{34}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of type super $\mathfrak{g}(1, 6)$ and another of type A_1 .

For $\begin{array}{c} \circ \\ \begin{array}{c} \overline{\zeta} \\ \downarrow \end{array} \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_1^2 = \mu_1; \quad y_{14} = \lambda_{14}; \quad y_{24} = 0; \quad y_{221} = 0; \\ y_4^2 = \mu_4; \quad y_{223} = 0; \quad y_{3332} = 0; \quad y_{3334} = \lambda_{3334}; \\ [y_2, y_{334}]_c + q_{34}[y_{(24)}, y_3]_c + (\zeta^2 - \zeta)q_{23}y_3y_{(24)} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of type super $\mathfrak{g}(1, 6)$ and another of type A_1 .

For $\begin{array}{c} \circ \\ \begin{array}{c} \overline{\zeta} \\ \downarrow \end{array} \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array} \begin{array}{c} \overline{\zeta} \\ \downarrow \\ \begin{array}{cc} \circ & \circ \end{array} \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_1^2 = \mu_1; \quad y_{13} = 0; \quad y_{14} = \lambda_{14}; \quad y_{24} = \lambda_{24}; \quad [[y_{(24)}, y_3]_c, y_3]_c = \lambda_{(24)}; \\ y_2^2 = \mu_2; \quad y_4^2 = \mu_4; \quad y_{332} = 0; \quad [y_{(13)}, y_2]_c = 0; \quad [y_{334}, y_{34}]_c = \lambda_{34}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 .

For $\begin{array}{c} \circ \\ \hline \zeta \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_2^2 = \mu_2; \quad y_{14} = \lambda_{14}; \quad y_{24} = \lambda_{24}; \quad [y_{(13)}, y_2]_c = 0; \\ y_4^2 = \mu_4; \quad y_{3332} = \lambda_{3332}; \quad y_{3334} = \lambda_{3334}; \quad y_1^2 = \mu_1; \end{aligned}$$

$$[y_2, y_{334}]_c + q_{34}[y_{(24)}, y_3]_c + (\zeta^2 - \zeta)q_{23}y_3y_{(24)} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 .

For $\begin{array}{c} \circ \\ \hline \zeta \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \zeta \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_2^2 = \mu_2; \quad y_{13} = 0; \quad y_{112} = 0; \quad [y_{332}, y_{32}]_c = \lambda_{32}; \\ y_4^2 = \mu_4; \quad y_{14} = 0; \quad y_{334} = 0; \quad [[y_{(24)}, y_3]_c, y_3]_c = \lambda_{(24)}; \\ y_{24} = \lambda_{24}; \quad [y_{(13)}, y_2]_c = 0; \quad [y_{3321}, y_{32}]_c = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 .

For $\begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \zeta \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \zeta \\ \hline \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{13} = 0; \quad y_3^2 = \mu_3; \quad [y_{(13)}, y_2]_c = 0; \quad y_{24} = \lambda_{24}; \quad [[y_{23}, y_{(24)}]_c, y_3]_c = \lambda_{234}; \\ y_{14} = 0; \quad y_4^2 = \mu_4; \quad y_{112} = 0; \quad y_2^2 = \mu_2; \quad [[y_{43}, y_{432}]_c, y_3]_c = \lambda_{432}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components, one of type super $\mathfrak{g}(2, 3)$ and another of type A_1 .

For $\begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \zeta \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \bar{\zeta} \\ \hline \circ \end{array} \begin{array}{c} \circ \\ \hline \zeta \\ \hline \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_{13} = 0; \quad y_{14} = 0; \quad y_{221} = \lambda_{221}; \quad y_{223} = \lambda_{223}; \quad y_{224} = 0; \\ y_{332} = \lambda_{332}; \quad y_{334} = 0; \quad y_{112} = \lambda_{112}; \quad y_4^2 = \mu_4; \end{aligned}$$

$$y_{(24)} - \zeta q_{34}[y_{24}, y_3]_c - (1 - \bar{\zeta})q_{23}y_3y_{24} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{24}, y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_3 and A_1 .

10.7. **Type $\mathfrak{g}(2, 6)$, $\zeta \in G'_3$.** The Weyl groupoid has four objects.

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \zeta$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{223} &= 0; & [[y_{(14)}, y_3]_c, y_2]_c, y_3]_c &= 0; \\ y_3^2 &= \mu_3; & y_{554} &= \lambda_{554}; & [[[y_{5432}, y_3]_c, y_4]_c, y_3]_c &= 0; \\ y_{221} &= \lambda_{221}; & y_{443} &= 0; & y_{445} &= \lambda_{225}; & y_{ij} &= \lambda_{ij}, \ i < j, \tilde{q}_{ij} = 1. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{23}, \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types A_3 and A_2 .

For $\begin{array}{c} \overset{-1}{\circ} \\ \zeta \downarrow \quad \searrow \zeta \\ \zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

with defining relations

$$\begin{aligned} y_{112} &= 0; & y_3^2 &= \mu_3; & y_{ij} &= \lambda_{ij}, \ i < j, \tilde{q}_{ij} = 1; & [y_{125}, y_2]_c &= 0; \\ y_{443} &= 0; & y_5^2 &= \mu_5; & [y_{(13)}, y_2]_c &= 0; & [y_{(24)}, y_3]_c &= 0; \\ y_2^2 &= \mu_2; & [y_{435}, y_3]_c &= 0; & y_{235} - q_{35}\bar{\zeta}[y_{25}, y_3]_c - q_{23}(1 - \zeta)y_3y_{25} &= 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \zeta \\ \downarrow \bar{\zeta} \\ \zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

with defining relations

$$\begin{aligned} y_{223} &= 0; & y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_3^2 &= \mu_3; & [y_{435}, y_3]_c &= 0; \\ [y_{(24)}, y_3]_c &= 0; & y_4^2 &= \mu_4; & y_{553} &= 0; & y_{ij} &= \lambda_{ij}, \ i < j, \tilde{q}_{ij} = 1. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{23}, \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types A_3 and A_2 .

For $\begin{array}{c} \zeta \\ \downarrow \bar{\zeta} \\ \zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

with defining relations

$$\begin{aligned} y_4^2 &= \mu_4; & y_{553} &= \lambda_{553}; & y_{112} &= \lambda_{112}; & y_{332} &= \lambda_{332}; & y_{221} &= \lambda_{221}; \\ y_{ij} &= \lambda_{ij}, \ i < j, \tilde{q}_{ij} = 1; & y_{335} &= \lambda_{335}; & y_{223} &= \lambda_{223}; & y_{334} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{34}, \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types A_4 and A_1 .

10.8. **Type $\mathfrak{el}(5; 3)$, $\zeta \in G'_3$.** The Weyl groupoid has fifteen objects.

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \bar{\zeta} \xrightarrow{\zeta} \bar{\zeta}^{-1}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{221} &= \lambda_{221}; & y_5^2 &= \mu_5; & y_{334} &= 0; & [[y_{(35)}, y_4]_c, y_4]_c &= 0; \\ y_{445} &= 0; & y_{112} &= \lambda_{112}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{223} &= \lambda_{223}; & y_{332} &= \lambda_{332}; & [y_{4432}, y_{43}]_c &= \lambda_{432}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types B_4 and A_1 .

For $\zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \bar{\zeta}^{-1} \xrightarrow{\bar{\zeta}} \bar{\zeta}^{-1}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_4^2 &= \mu_4; & y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & [[y_{54}, y_{543}]_c, y_4]_c &= 0; \\ y_{334} &= 0; & y_5^2 &= \mu_5; & y_{223} &= \lambda_{223}; & y_{332} &= \lambda_{332}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types A_4 and A_1 .

For $\bar{\zeta}^{-1} \xrightarrow{\zeta} \bar{\zeta}^{-1} \xrightarrow{\bar{\zeta}} \bar{\zeta}^{-1} \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \zeta$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{443} &= 0; & y_{445} &= \lambda_{445}; & [[y_{23}, y_{(24)}]_c, y_3]_c &= 0; & y_1^2 &= \mu_1; & y_2^2 &= \mu_2 \\ y_3^2 &= \mu_3; & y_{554} &= \lambda_{554}; & [y_{(13)}, y_2]_c &= \nu_2; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types A_3 and A_2 .

For $\zeta \xrightarrow{\bar{\zeta}} \bar{\zeta}^{-1} \xrightarrow{\zeta} \bar{\zeta}^{-1} \xrightarrow{\bar{\zeta}} \bar{\zeta}^{-1} \xrightarrow{\zeta} \bar{\zeta}^{-1} \xrightarrow{\zeta}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations $y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1$;

$$\begin{aligned} y_2^2 &= \mu_2; & y_{112} &= 0; & y_{445} &= \lambda_{445}; & [[y_{(24)}, y_3]_c, y_3]_c &= 0; \\ y_{332} &= 0; & y_{443} &= 0; & y_{554} &= \lambda_{554}; & [y_{3345}, y_{34}]_c &= \lambda_{345}; & [y_{(13)}, y_2]_c &= \nu_2. \end{aligned}$$

If $\nu_2 = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types B_3 and A_2 .

If $\nu_2 \neq 0$, then $\lambda_{345} = 0$, so $\tilde{\mathcal{E}}_q / \langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types A_3 and A_2 .

For $\bar{\zeta}^{-1} \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \bar{\zeta}^{-1} \xrightarrow{\bar{\zeta}} \zeta \xrightarrow{\bar{\zeta}} \zeta$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_5$, with defining relations $y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1$;

$$\begin{aligned} y_3^2 &= \mu_3; & y_{445} &= \lambda_{445}; & y_{221} &= 0; & [[[y_{5432}, y_3]_c, y_4]_c, y_3]_c &= 0; \\ y_{223} &= 0; & y_1^2 &= \mu_1; & y_{554} &= \lambda_{554}; & y_{443} &= 0; & [[[y_{(14)}, y_3]_c, y_2]_c, y_3]_c &= \lambda_{(14)}. \end{aligned}$$

If $\lambda_{(14)} = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types A_3, A_2 .

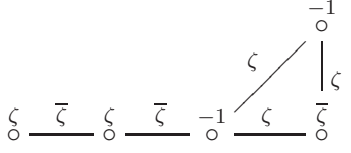
If $\lambda_{(14)} \neq 0$ but all the other λ 's are zero, then $\tilde{\mathcal{E}}_q / \langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, \mathfrak{q}'$ with two components of types B_4, A_1 . The general case follows by Lemma 3.3.

For  the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}$, $i < j$, $\tilde{q}_{ij} = 1$;

$$\begin{aligned} y_2^2 &= \mu_2; & [y_{125}, y_2]_c &= \nu'_2; & y_3^2 &= \mu_3; & [y_{(24)}, y_3]_c &= 0; \\ [y_{(13)}, y_2]_c &= \nu_2; & y_5^2 &= \mu_5; & y_1^2 &= \mu_1; & [y_{435}, y_3]_c &= 0; \\ y_{443} &= 0; & y_{235} - q_{35}\bar{\zeta}[y_{25}, y_3]_c - q_{23}(1 - \zeta)y_3y_{25} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super CD and A_1 .

For  the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}$, $i < j$, $\tilde{q}_{ij} = 1$;

$$\begin{aligned} y_3^2 &= \mu_3; & [y_{(24)}, y_3]_c &= \nu_3; & y_{445} &= 0; & y_{112} &= \lambda_{112}; \\ y_{221} &= \lambda_{221}; & y_{223} &= 0; & y_{443} &= 0; & [y_{235}, y_3]_c &= 0; \\ y_5^2 &= \mu_5; & y_{(35)} - q_{45}\bar{\zeta}[y_{35}, y_4]_c - q_{34}(1 - \zeta)y_4y_{35} &= 0. \end{aligned}$$

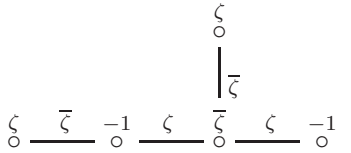
Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{35}, y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

For  the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}$, $i < j$, $\tilde{q}_{ij} = 1$;

$$\begin{aligned} y_1^2 &= \mu_1; & [y_{(24)}, y_3]_c &= \nu_2; & y_3^2 &= \mu_3; & y_{221} &= 0; \\ y_{223} &= 0; & y_5^2 &= \mu_5; & y_{443} &= 0; & [y_{435}, y_3]_c &= 0; \\ y_{225} &= 0; & y_{235} - q_{35}\bar{\zeta}[y_{25}, y_3]_c - q_{23}(1 - \zeta)y_3y_{25} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{25}, y_{35} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

For  the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}$, $i < j$, $\tilde{q}_{ij} = 1$;

$$\begin{aligned} y_{112} &= 0; & y_2^2 &= \mu_2; & [y_{(13)}, y_2]_c &= \nu_2; & y_4^2 &= \mu_4; & [[y_{235}, y_3]_c, y_3]_c &= 0; \\ y_{553} &= 0; & y_{332} &= 0; & y_{334} &= 0; & [[y_{435}, y_3]_c, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{35} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types super A_4 and A_1 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \\ \circ \text{---} \zeta \text{---} \circ \text{---} \bar{\zeta} \text{---} \circ \text{---} \zeta \text{---} \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1$;

$$\begin{aligned} y_2^2 &= \mu_2; & [[y_{23}, y_{235}]_c, y_3]_c &= 0; & y_3^2 &= \mu_3; & [y_{(13)}, y_2]_c &= \nu_2; \\ y_4^2 &= \mu_4; & y_{553} &= 0; & [y_{(24)}, y_3]_c &= \nu_3; & y_1^2 &= \mu_1; & [y_{435}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{35} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types super A_4 and A_1 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \\ \circ \text{---} \bar{\zeta} \text{---} \zeta \text{---} \circ \text{---} \bar{\zeta} \text{---} \circ \text{---} \zeta \text{---} \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1$;

$$\begin{aligned} y_1^2 &= \mu_1; & y_3^2 &= \mu_3; & y_{223} &= 0; & [[[y_{1235}, y_3]_c, y_2]_c, y_3]_c &= \lambda_{1235}; \\ y_{221} &= 0; & [y_{(24)}, y_3]_c &= 0; & y_4^2 &= \mu_4; & y_{553} &= 0; & [y_{435}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types super CD and A_1 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \\ \circ \text{---} \zeta \text{---} \circ \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1$;

$$\begin{aligned} y_1^2 &= \mu_1; & [y_{(13)}, y_2]_c &= 0; & y_2^2 &= \mu_2; & y_{335} &= \lambda_{335}; \\ y_{332} &= 0; & y_4^2 &= \mu_4; & y_{334} &= 0; & y_{553} &= \lambda_{553}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types super A_4 and A_1 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \\ \zeta \text{---} \bar{\zeta} \text{---} \circ \text{---} \zeta \text{---} \circ \text{---} \bar{\zeta} \text{---} \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_5$,

with defining relations $y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1$;

$$\begin{aligned} y_2^2 &= \mu_2; & [[y_{43}, y_{435}]_c, y_3]_c &= 0; & y_3^2 &= \mu_3; & [y_{(24)}, y_3]_c &= \nu_3; \\ y_{112} &= 0; & y_4^2 &= \mu_4; & [y_{(13)}, y_2]_c &= 0; & y_{553} &= 0; & [y_{235}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{35} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types super A_4 and A_1 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$, $\begin{array}{ccccccc} & & \zeta & & & & \\ & & | & & & & \\ -1 & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & -1 \\ \circ & & \circ & & \circ & & \circ \end{array}$

$$\begin{array}{llll} y_1^2 = \mu_1; & y_{221} = 0; & y_{553} = \lambda_{553}; & y_{332} = \lambda_{332}; \\ y_4^2 = \mu_4; & y_{223} = \lambda_{223}; & y_{334} = 0; & y_{335} = \lambda_{335}. \end{array}$$

For $\begin{array}{c} \zeta \\ | \overline{\zeta} \\ \zeta \text{---} \zeta \text{---} \overline{1} \text{---} \overline{\zeta} \text{---} \zeta \text{---} \overline{\zeta} \text{---} \zeta \end{array}$ the algebra $\widetilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\begin{aligned} y_{552} &= 0; & [y_{1(3)}, y_2]_c &= \nu_2; & y_{443} &= \lambda_{443}; & [[[y_{4325}, y_2]_c, y_3]_c, y_2]_c &= 0; \\ y_{112} &= 0; & [y_{125}, y_2]_c &= \nu'_2; & y_{334} &= \lambda_{334}; & y_{332} &= 0; & y_2^2 &= \mu_2. \end{aligned}$$

10.9. **Type $\mathfrak{g}(8, 3)$, $\zeta \in G'_3$.** The Weyl groupoid has 21 objects:

For $\overset{-1}{\underset{\circ}{\text{---}}}\overset{\zeta}{\underset{\circ}{\text{---}}}\overset{\bar{\zeta}}{\underset{\circ}{\text{---}}}\overset{\zeta}{\underset{\circ}{\text{---}}}\overset{\bar{\zeta}}{\underset{\circ}{\text{---}}}\overset{\zeta}{\underset{\circ}{\text{---}}}\overset{\bar{\zeta}}{\underset{\circ}{\text{---}}}\overset{\zeta}{\underset{\circ}{\text{---}}}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in$
, with defining relations

$$\begin{array}{llll} y_{221} = 0; & y_{223} = \lambda_{223}; & y_{332} = \lambda_{332}; & y_{ij} = \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_{443} = 0; & y_{445} = \lambda_{445}; & [y_{3345}, y_{34}]_c = \lambda_{345}; & \\ y_{554} = \lambda_{554}; & y_1^2 = \mu_1; & [[y_{(24)}, y_3]_c, y_3]_c = \nu_3; & \end{array}$$

For $\begin{smallmatrix} & -1 \\ \circ & \end{smallmatrix} \xrightarrow{\bar{\zeta}} \begin{smallmatrix} & \zeta \\ \circ & \end{smallmatrix} \xrightarrow{\bar{\zeta}} \begin{smallmatrix} & \zeta \\ \circ & \end{smallmatrix} \xrightarrow{\bar{\zeta}} \begin{smallmatrix} & \zeta \\ \circ & \end{smallmatrix} \xrightarrow{-1} \begin{smallmatrix} & -1 \\ \circ & \end{smallmatrix}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{array}{llll} y_{221} = 0; & y_{223} = \lambda_{223}; & y_{332} = \lambda_{332}; & y_{ij} = \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_{334} = 0; & y_{445} = 0; & [y_{4432}, y_{43}]_c = \lambda_{432}; & \\ y_1^2 = \mu_1; & y_5^2 = \mu_5; & [[y_{(35)}, y_4]_c, y_4]_c = 0; & . \end{array}$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components of types super B_4 and A_1 .

For $\overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{\bar{\zeta}}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} [y_{(13)}, y_2]_c &= 0; & y_{332} &= 0; & y_{334} &= 0; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_{445} &= 0; & y_1^2 &= \mu_1; & [y_{4432}, y_{43}]_c &= 0; \\ y_2^2 &= \mu_2; & y_5^2 &= \mu_5; & [[y_{(35)}, y_4]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super B_4 and A_1 .

For $\overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{221} &= 0; & y_{223} &= \lambda_{223}; & y_{334} &= 0; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_5^2 &= \mu_5; & y_{332} &= \lambda_{332}; & y_1^2 &= \mu_1; & [[y_{54}, y_{543}]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_3 and A_2 .

For $\overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{\bar{\zeta}}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{112} &= 0; & [y_{443}, y_{43}]_c &= \lambda_{43}; & y_{445} &= 0; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ [y_{(13)}, y_2]_c &= 0; & y_2^2 &= \mu_2; & [y_{4432}, y_{43}]_c &= 0; \\ y_5^2 &= \mu_5; & [y_{(24)}, y_3]_c &= 0; & y_3^2 &= \mu_3; & [[y_{(35)}, y_4]_c, y_4]_c &= \lambda_{(35)}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_2 and $\mathfrak{g}(2, 3)$.

For $\overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_5^2 &= \mu_5; & y_{332} &= 0; & y_{334} &= 0; & y_1^2 &= \mu_1; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ [y_{(13)}, y_2]_c &= 0; & y_2^2 &= \mu_2; & y_4^2 &= \mu_4; & [[y_{54}, y_{543}]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_2 and A_3 .

For $\overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{\zeta}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-\zeta}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & y_{221} &= \lambda_{221}; & y_{4443} &= \lambda_{4443}; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_5^2 &= \mu_5; & y_{112} &= \lambda_{112}; & y_{4445} &= \lambda_{4445}; & [y_{(24)}, y_3]_c &= 0; \\ y_{223} &= 0; & [y_3, y_{445}]_c &+ q_{45}[y_{(35)}, y_4]_c + (\zeta - \bar{\zeta})q_{34}y_4y_{(35)} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_2 and $\mathfrak{g}(2, 3)$.

For $\begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & [y_{(24)}, y_3]_c &= \nu_3; & y_{112} &= \lambda_{112}; & y_{223} &= 0; \\ y_5^2 &= \mu_5; & [y_{445}, y_{45}]_c &= \lambda_{45}; & y_{221} &= \lambda_{221}; & y_{445} &= 0; \\ y_{ij} &= \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1; & [[y_{(35)}, y_4]_c, y_4]_c &= \lambda_{(35)}. \end{aligned}$$

If $\nu_3 = 0$, then $\tilde{\mathcal{E}}_q / \langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of types super A_2 and $\mathfrak{g}(2, 3)$. If $\nu_3 \neq 0$, then $\lambda_{(35)} = \lambda_{45} = 0$, so $\tilde{\mathcal{E}}_q / \langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of types super A_4 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_4^2 &= \mu_4; & y_5^2 &= \mu_5; & [y_{(24)}, y_3]_c &= \nu_3; & y_{ij} &= \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; & [y_{(13)}, y_2]_c &= 0; & [[y_{34}, y_{(35)}]_c, y_4]_c &= \lambda_{345}; \\ y_3^2 &= \mu_3; & y_{112} &= 0; & [[y_{54}, y_{543}]_c, y_4]_c &= \lambda_{543}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of types A_1 and $\mathfrak{g}(4, 3)$.

For $\begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} [y_{(13)}, y_2]_c &= 0; & y_2^2 &= \mu_2; & y_{112} &= 0; & y_{445} &= \lambda_{445}; & y_{ij} &= \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_3^2 &= \mu_3; & y_{443} &= 0; & y_{554} &= \lambda_{554}; & [[y_{23}, y_{(24)}]_c, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of types super A_2 and A_3 .

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & [y_{(13)}, y_2]_c &= 0; & y_{332} &= 0; & y_{445} &= \lambda_{445}; \\ y_2^2 &= \mu_2; & [y_{3345}, y_{34}]_c &= \lambda'_{(35)}; & y_{443} &= 0; & y_{554} &= \lambda_{554}; \\ y_{ij} &= \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1; & [[y_{(35)}, y_4]_c, y_4]_c &= \lambda_{(35)}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{q'}$, q' with two components of types super A_2 and C_3 .

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$y_1^2 = \mu_1; \quad [y_{(24)}, y_3]_c = \nu_3; \quad y_{221} = 0; \quad y_{223} = 0; \quad y_{ij} = \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1;$$

$$y_3^2 = \mu_3; \quad y_5^2 = \mu_5; \quad y_{443} = 0; \quad y_{445} = 0; \quad [y_{235}, y_3]_c = 0;$$

$$y_{(35)} - q_{45}\bar{\zeta}[y_{35}, y_4]_c - q_{34}(1 - \zeta)y_4y_{35} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34}, y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

For $\begin{array}{c} \overset{-1}{\circ} \\ \downarrow \zeta \quad \searrow \zeta \\ \overset{-1}{\circ} \quad \zeta \quad \overset{-1}{\circ} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$$

with defining relations

$$[y_{(13)}, y_2]_c = \nu_2; \quad y_{443} = 0; \quad y_1^2 = \mu_1; \quad y_{ij} = \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1;$$

$$[y_{(24)}, y_3]_c = 0; \quad y_{445} = 0; \quad y_2^2 = \mu_2; \quad [y_{235}, y_3]_c = \nu'_3;$$

$$y_3^2 = \mu_3; \quad y_5^2 = \mu_5; \quad y_{(35)} - q_{45}\bar{\zeta}[y_{35}, y_4]_c - q_{34}(1 - \zeta)y_4y_{35} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34}, y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

For $\begin{array}{c} \overset{-1}{\circ} \\ \downarrow \zeta \quad \searrow \zeta \\ \overset{-1}{\circ} \quad \zeta \quad \overset{-1}{\circ} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$$

with defining relations

$$[y_{(24)}, y_3]_c = \nu_3; \quad y_{112} = \lambda_{112}; \quad y_{223} = 0; \quad y_{ij} = \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1;$$

$$y_{443} = 0; \quad y_{221} = \lambda_{221}; \quad y_{225} = 0; \quad [y_{435}, y_3]_c = 0;$$

$$y_3^2 = \mu_3; \quad y_5^2 = \mu_5; \quad y_{235} - q_{35}\bar{\zeta}[y_{25}, y_3]_c - q_{23}(1 - \zeta)y_3y_{25} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{25}, y_{35} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

For $\begin{array}{c} \overset{-1}{\circ} \\ \downarrow \zeta \quad \searrow \zeta \\ \overset{-1}{\circ} \quad \zeta \quad \overset{-1}{\circ} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ} \xrightarrow{\bar{\zeta}} \overset{-1}{\circ} \xrightarrow{\zeta} \overset{-1}{\circ}$$

with defining relations

$$[y_{(13)}, y_2]_c = \nu_2; \quad y_{112} = 0; \quad y_{332} = 0; \quad y_{ij} = \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1;$$

$$y_{445} = 0; \quad y_{334} = \lambda_{334}; \quad y_{335} = 0; \quad y_{443} = \lambda_{443};$$

$$y_2^2 = \mu_2; \quad y_5^2 = \mu_5; \quad y_{(35)} - q_{45}\bar{\zeta}[y_{35}, y_4]_c - q_{34}(1 - \zeta)y_4y_{35} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{35}, y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

For

$$\begin{array}{ccccc}
 & & \zeta & & \\
 & & | & & \\
 & & \bar{\zeta} & & \\
 \ominus^{-1} & \xrightarrow{\zeta} & \ominus^{-1} & \xrightarrow{\zeta} & \ominus^{-1} \\
 & & \bar{\zeta} & & \\
 \ominus^{-1} & \xrightarrow{\zeta} & \ominus^{-1} & \xrightarrow{\zeta} & \ominus^{-1}
 \end{array}$$

the algebra $\widetilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

$$y_3^2 = \mu_3; \quad [y_{(24)}, y_3]_c = 0; \quad y_{221} = 0; \quad y_{223} = 0; \quad y_{ij} = \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_4^2 = \mu_4; \quad [y_{235}, y_3]_c = \nu_3'; \quad y_{553} = 0; \quad y_1^2 = \mu_1; \quad [[y_{43}, y_{435}]_c, y_3]_c = 0.$$

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$, $\begin{array}{ccccccc} \bar{\zeta} & \zeta & -1 & \bar{\zeta} & \zeta & \bar{\zeta} & -1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$

$$y_{112} = 0; \quad y_2^2 = \mu_2; \quad y_{332} = 0; \quad y_{335} = \lambda_{335}; \quad y_{ij} = \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_4^2 = \mu_4; \quad y_{334} = 0; \quad y_{553} = \lambda_{553}; \quad [y_{(13)}, y_2]_c = \nu_2.$$

For

$$\begin{array}{ccccccc}
 & & & & \zeta & & \\
 & & & & | & & \\
 & & & & \bar{\zeta} & & \\
 \circ^{-1} & \xrightarrow{\bar{\zeta}} & \circ^{-1} & \xrightarrow{\zeta} & \circ^{-1} & \xrightarrow{\zeta} & \circ^{-1}
 \end{array}$$

the algebra $\widetilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\begin{aligned} y_1^2 &= \mu_1; \quad [[y_{235}, y_3]_c, y_3]_c = 0; \quad [y_{(13)}, y_2]_c = 0; \quad y_{ij} = \lambda_{ij}, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; \quad [[y_{435}, y_3]_c, y_3]_c = 0; \quad y_4^2 = \mu_4; \quad y_{332} = y_{334} = y_{553} = 0. \end{aligned}$$

For $\begin{array}{c} \zeta \\ \circ \\ | \bar{\zeta} \\ \circ \\ \zeta \quad \bar{\zeta} \quad \zeta \\ \circ \quad \circ \quad \circ \\ \zeta \quad \zeta \quad \zeta \\ \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\begin{aligned} y_1^2 &= \mu_1; & y_{221} &= y_{334} = y_{553} = 0; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & [[y_{435}, y_3]_c, y_3]_c &= 0; & y_{332} &= \lambda_{332}; & [[y_{235}, y_3]_c, y_3]_c &= \lambda_{235}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super CD and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_5$,
 $\begin{array}{c} \bar{\zeta} \quad \zeta \quad -1 \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$

with defining relations

$$\begin{aligned} y_{112} &= 0; & [y_{(24)}, y_3]_c &= \nu_3; & y_4^2 &= \mu_4; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ [y_{(13)}, y_2]_c &= 0; & y_3^2 &= \mu_3; & [[y_{23}, y_{235}]_c, y_3]_c &= 0; \\ y_{553} &= 0; & y_2^2 &= \mu_2; & [y_{435}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{35} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_5$,
 $\begin{array}{c} -1 \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$

with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & [y_{(24)}, y_3]_c &= \nu_3; & y_{553} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ [y_{(13)}, y_2]_c &= \nu_2; & y_4^2 &= \mu_4; & [[y_{43}, y_{435}]_c, y_3]_c &= 0; \\ y_3^2 &= \mu_3; & y_2^2 &= \mu_2; & [y_{235}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{35} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types super A_4 and A_1 .

10.10. **Type $\mathfrak{g}(4, 6)$** , $\zeta \in G'_3$. The Weyl groupoid has seven objects.

For $\begin{array}{c} \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{665} &= \lambda_{665}; & y_{332} &= \lambda_{332}; & y_{334} &= 0; & [[[y_{(25)}, y_4]_c, y_3]_c, y_4]_c &= 0; \\ y_4^2 &= \mu_4; & y_{554} &= 0; & y_{556} &= \lambda_{556}; & [[y_{6543}, y_4]_c, y_5]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_3 .

For $\begin{array}{c} \zeta \\ \circ \\ \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_6$,
 $\begin{array}{c} \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$

\mathbb{I}_6 , with defining relations

$$\begin{aligned} y_4^2 &= \mu_4; & y_{221} &= \lambda_{221}; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_5^2 &= \mu_5; & y_{334} &= 0; & y_{664} &= 0; & [y_{546}, y_4]_c &= 0; \\ [y_{(35)}, y_4]_c &= 0; & y_{112} &= \lambda_{112}; & y_{332} &= \lambda_{332}; & [[[y_{2346}, y_4]_c, y_3]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}, \mathbf{q}'$ with two components of types CD and A_1 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in$

$$\begin{array}{ccccccc} \zeta & \bar{\zeta} & -1 & \zeta & -1 & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$$

\mathbb{I}_6 , with defining relations

$$\begin{aligned} [y_{(13)}, y_2]_c &= 0; & y_{445} &= \lambda_{445}; & y_{443} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{112} &= 0; & y_{554} &= \lambda_{554}; & y_{663} &= 0; & [y_{236}, y_3]_c &= 0; \\ [y_{(24)}, y_3]_c &= 0; & y_2^2 &= \mu_2; & y_3^2 &= \mu_3; & [[y_{5436}, y_3]_c, y_4]_c, y_3 &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}, \mathbf{q}'$ with two components of types A_4 and A_2 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in \mathbb{I}_6$,

$$\begin{array}{ccccccc} \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & -1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$$

with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{332} &= \lambda_{332}; & y_{334} &= \lambda_{334}; & y_{443} &= \lambda_{443}; & y_{445} &= 0; \\ y_{446} &= \lambda_{446}; & y_{664} &= \lambda_{664}; & y_5^2 &= \mu_5. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}, \mathbf{q}'$ with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in$

$$\begin{array}{ccccccc} -1 & \zeta & -1 & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$$

\mathbb{I}_6 , with defining relations

$$\begin{aligned} y_{332} &= 0; & y_{334} &= \lambda_{334}; & y_{336} &= \lambda_{336}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{663} &= \lambda_{663}; & y_{443} &= \lambda_{443}; & y_{445} &= \lambda_{443}; & [y_{(13)}, y_2]_c &= 0; \\ y_{554} &= \lambda_{554}; & y_1^2 &= \mu_1; & y_2^2 &= \mu_2. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}, \mathbf{q}'$ with two components of types A_2 and A_4 .

For $\begin{array}{c} \zeta \\ | \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by $y_i, i \in$

$$\begin{array}{ccccccc} -1 & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & -1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$$

\mathbb{I}_6 , with defining relations

$$y_{221} = 0; \quad y_{223} = \lambda_{223}; \quad y_{332} = \lambda_{332}; \quad y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1;$$

$$\begin{aligned} y_{334} &= \lambda_{334}; & y_{443} &= \lambda_{443}; & y_{445} &= 0; & y_{336} &= \lambda_{336}; \\ y_{663} &= \lambda_{663}; & y_1^2 &= \mu_1; & y_5^2 &= \mu_5. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types CD and A_1 .

For $\begin{array}{c} \circ^{-1} \\ \mid \searrow \zeta \\ \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \circ^{-1} \quad \zeta \quad \bar{\zeta} \quad \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in$

\mathbb{I}_6 , with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_6^2 &= \mu_6; & y_{223} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{554} &= 0; & y_4^2 &= \mu_4; & [y_{(24)}, y_3]_c &= 0; & [y_{236}, y_3]_c &= 0; \\ y_{221} &= \lambda_{221}; & y_3^2 &= \mu_3; & [y_{(35)}, y_4]_c &= 0; & [y_{546}, y_4]_c &= 0; \\ y_{346} - q_{46}\bar{\zeta}[y_{36}, y_4]_c - q_{34}(1 - \zeta)y_4y_{36} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{36}, y_{46} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_5 and A_1 .

10.11. **Type $\mathfrak{g}(6, 6)$** , $\zeta \in G'_3$. The Weyl groupoid has 21 objects.

For $\begin{array}{c} \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \bar{\zeta} \quad \zeta \quad \circ^{-1} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{556} &= 0 & y_{332} &= \lambda_{332}; & y_{334} &= \lambda_{334}; & [[y_{(46)}, y_5]_c, y_5]_c &= 0; \\ y_6^2 &= \mu_6; & y_{443} &= \lambda_{443}; & y_{445} &= \lambda_{445}; & [y_{5543}, y_{54}]_c &= \lambda_{543}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{56} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types B_5 and A_1 .

For $\begin{array}{c} \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \circ^{-1} \quad \bar{\zeta} \quad \circ^{-1} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_5^2 &= \mu_5; & y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_6^2 &= \mu_6; & y_{223} &= \lambda_{223}; & y_{332} &= \lambda_{332}; & [[y_{65}, y_{654}]_c, y_5]_c &= 0; \\ y_{334} &= \lambda_{334}; & y_{443} &= \lambda_{443}; & y_{445} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{56} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \quad \bar{\zeta} \quad \circ^{-1} \quad \zeta \quad \circ^{-1} \quad \bar{\zeta} \quad \circ^{-1} \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_2^2 &= \mu_2; & y_{554} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{556} &= \lambda_{556}; & y_3^2 &= \mu_3; & y_{665} &= \lambda_{665}; & [[y_{34}, y_{(35)}]_c, y_4]_c &= 0; \\ y_4^2 &= \mu_4; & [y_{(13)}, y_2]_c &= 0; & [y_{(24)}, y_3]_c &= \nu_3. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_4 and A_2 .

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & y_{334} &= 0; & y_{556} &= \lambda_{556}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; & y_{554} &= 0; & y_{665} &= \lambda_{665}; & [[y_{(25)}, y_4]_c, y_3]_c, y_4]_c &= \lambda_{(25)}; \\ y_4^2 &= \mu_4; & y_{332} &= 0; & [y_{(13)}, y_2]_c &= 0; & [[y_{6543}, y_4]_c, y_5]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types $\mathfrak{sl}(5, 3)$ and A_1 .

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{221} &= 0; & y_{223} &= \lambda_{223}; & y_{332} &= \lambda_{332}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_1^2 &= \mu_1; & y_{334} &= 0; & y_{554} &= 0; & [[y_{(25)}, y_4]_c, y_3]_c, y_4]_c &= 0; \\ y_4^2 &= \mu_4; & y_{556} &= \lambda_{556}; & y_{665} &= \lambda_{665}; & [[y_{6543}, y_4]_c, y_5]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_4 and A_2 .

For $\begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{665} &= \lambda_{665}; & y_{223} &= 0; & y_{443} &= 0; & [[y_{(35)}, y_4]_c, y_4]_c &= 0; \\ [y_{(24)}, y_3]_c &= \nu_2; & y_{554} &= 0; & y_{556} &= \lambda_{665}; & [y_{4456}, y_{45}]_c &= \lambda_{456}. \end{aligned}$$

If $\nu_2 = 0$, then $\tilde{\mathcal{E}}_q/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_3 and B_3 .

If $\nu_2 \neq 0$, then $\lambda_{456} = 0$, so $\tilde{\mathcal{E}}_q/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_4 and A_2 .

For $\begin{array}{c} -1 \\ \circ \end{array} \begin{array}{c} \zeta \\ \downarrow \end{array} \begin{array}{c} \zeta \\ \downarrow \end{array} \begin{array}{c} \zeta \\ \downarrow \end{array} \begin{array}{c} \zeta \\ \downarrow \end{array} \begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_{332} &= \lambda_{332}; & y_{334} &= 0; & [y_{346}, y_4]_c &= 0; \\ y_5^2 &= \mu_5; & y_{554} &= 0; & y_{556} &= 0; & [y_{(35)}, y_4]_c &= \nu_4; \\ y_{(46)} - q_{56}\bar{\zeta}[y_{46}, y_5]_c - q_{45}(1 - \zeta)y_5y_{46} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q/\langle y_{46}, y_{56} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_5 and A_1 .

For  the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{221} &= 0; & y_{223} &= 0; & y_{554} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_1^2 &= \mu_1; & [y_{(24)}, y_3]_c &= 0; & [y_{236}, y_3]_c &= 0; \\ y_6^2 &= \mu_6; & y_3^2 &= \mu_3; & [y_{(35)}, y_4]_c &= 0; & [y_{546}, y_4]_c &= 0; \end{aligned}$$

$$y_{346} - q_{46}\bar{\zeta}[y_{36}, y_4]_c - q_{34}(1 - \zeta)y_4y_{36} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{36}, y_{46} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components of types A_5 and A_1 .

For  the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{554} &= 0; & y_1^2 &= \mu_1; & [y_{(13)}, y_2]_c &= \nu_2; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_2^2 &= \mu_2; & [y_{(24)}, y_3]_c &= \nu_3; & [y_{236}, y_3]_c &= \nu'_3; \\ y_6^2 &= \mu_6; & y_3^2 &= \mu_3; & [y_{(35)}, y_4]_c &= 0; & [y_{546}, y_4]_c &= 0; \end{aligned}$$

$$y_{346} - q_{46}\bar{\zeta}[y_{36}, y_4]_c - q_{34}(1 - \zeta)y_4y_{36} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components of types CD and A_1 .

For  the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{332} &= 0; & y_{334} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_{336} &= 0; & [y_{(13)}, y_2]_c &= \nu_2; & [y_{(35)}, y_4]_c &= \nu_4; \\ y_6^2 &= \mu_6; & y_{554} &= 0; & y_2^2 &= \mu_2; & [y_{546}, y_4]_c &= 0; \end{aligned}$$

$$y_{346} - q_{46}\bar{\zeta}[y_{36}, y_4]_c - q_{34}(1 - \zeta)y_4y_{36} = 0.$$

Here, $\tilde{\mathcal{E}}_{\mathfrak{q}}/\langle y_{36}, y_{46} \rangle \simeq \tilde{\mathcal{E}}_{\mathfrak{q}'}$, \mathfrak{q}' with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \hline \bar{\zeta} \\ \hline \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= 0; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_{664} &= 0; & [y_{(24)}, y_3]_c &= 0; & [[y_{54}, y_{546}]_c, y_4]_c &= 0; \\ y_5^2 &= \mu_5; & y_3^2 &= \mu_3; & [y_{(35)}, y_4]_c &= \nu_4; & [y_{346}, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{46} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \hline \bar{\zeta} \\ \hline \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_3^2 &= \mu_3; & y_{112} &= 0; & y_{443} &= 0; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_5^2 &= \mu_5; & y_{445} &= 0; & y_{446} &= \lambda_{446}; & [y_{(13)}, y_2]_c &= 0; \\ & & y_{664} &= \lambda_{664}; & y_2^2 &= \mu_2; & [y_{(24)}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of type A_3 .

For $\begin{array}{c} \zeta \\ \circ \\ \hline \bar{\zeta} \\ \hline \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad \bar{\zeta} \quad \zeta \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= 0; & y_{ij} &= \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_3^2 &= \mu_3; & y_{443} &= 0; & [y_{(24)}, y_3]_c &= \nu_3; & [[y_{346}, y_4]_c, y_4]_c &= 0; \\ y_5^2 &= \mu_5; & y_{445} &= 0; & y_{664} &= 0; & [[y_{546}, y_4]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{46} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \hline \bar{\zeta} \\ \hline -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$y_1^2 = \mu_1; \quad y_{332} = 0; \quad y_{334} = \lambda_{334}; \quad y_{ij} = \lambda_{ij}, \ i < j, \ \tilde{q}_{ij} = 1;$$

$$\begin{aligned} y_2^2 &= \mu_2; & y_{443} &= \lambda_{443}; & y_{445} &= 0; & [y_{(13)}, y_2]_c &= 0; \\ y_5^2 &= \mu_5; & y_{446} &= \lambda_{446}; & y_{664} &= \lambda_{664}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \mid \bar{\zeta} \\ \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{664} &= 0; & [y_{(13)}, y_2]_c &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_2^2 &= \mu_2; & [y_{(24)}, y_3]_c &= \nu_3; & [[y_{34}, y_{346}]_c, y_4]_c &= 0; \\ y_5^2 &= \mu_5; & y_3^2 &= \mu_3; & [y_{(35)}, y_4]_c &= \nu_4; & [y_{546}, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{46} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \mid \bar{\zeta} \\ -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & y_{221} &= 0; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_5^2 &= \mu_5; & y_{332} &= \lambda_{332}; & y_{334} &= \lambda_{334}; & y_{443} &= \lambda_{443}; \\ & & y_{445} &= 0; & y_{446} &= \lambda_{446}; & y_{664} &= \lambda_{664}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \mid \bar{\zeta} \\ -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & y_{332} &= 0; & y_{334} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; & y_{664} &= 0; & [y_{(13)}, y_2]_c &= 0; & [y_{564}, y_4]_c &= 0; \\ & & y_4^2 &= \mu_4; & y_5^2 &= \mu_5; & [y_{(35)}, y_4]_c &= 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \zeta \\ \circ \\ \mid \bar{\zeta} \\ \bar{\zeta} \text{ --- } \zeta \text{ --- } \overset{-1}{\circ} \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in$

\mathbb{I}_6 , with defining relations

$$\begin{aligned} y_2^2 &= \mu_2; & y_{112} &= 0; & y_{332} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{554} &= \lambda_{554}; & y_{334} &= \lambda_{334}; & y_{336} &= \lambda_{336}; & [y_{(13)}, y_2]_c &= \nu_2; \\ & & y_{663} &= \lambda_{663}; & y_{443} &= \lambda_{443}; & y_{445} &= \lambda_{445}. \end{aligned}$$

If $\nu_2 = 0$, then $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_2 and A_4 .
If $\nu_2 \neq 0$, then $\lambda_{336} = \lambda_{663} = 0$, so $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{36} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_5 and A_1 .

For $\begin{array}{c} \zeta \\ \circ \\ \mid \bar{\zeta} \\ -1 \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } -1 \text{ --- } \zeta \text{ --- } -1 \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & y_{223} &= \lambda_{223}; & y_{334} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \mu_4; & y_{332} &= \lambda_{332}; & [y_{(35)}, y_4]_c &= 0; & [[y_{2346}, y_4]_c, y_3]_c, y_4]_c &= 0; \\ y_5^2 &= \mu_5; & y_{664} &= 0; & y_{221} &= 0; & [y_{546}, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of type A_3 .

For $\begin{array}{c} \zeta \\ \circ \\ \mid \bar{\zeta} \\ -1 \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } -1 \text{ --- } \bar{\zeta} \text{ --- } \zeta \text{ --- } \bar{\zeta} \text{ --- } \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{221} &= 0; & y_{223} &= 0; & y_{443} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_1^2 &= \mu_1; & y_{445} &= \lambda_{445}; & [y_{(24)}, y_3]_c &= 0; & [[y_{5436}, y_3]_c, y_4]_c, y_3]_c &= 0; \\ y_3^2 &= \mu_3; & y_{554} &= \lambda_{554}; & y_{663} &= 0; & [y_{236}, y_3]_c &= \nu'_3. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_4 and A_2 .

For $\begin{array}{c} \circ \\ \zeta \\ \mid \\ \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,
 $\begin{array}{ccccccc} \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\zeta} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\zeta} & \circ \end{array}$

$i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & y_{445} &= \lambda_{445}; & y_{554} &= \lambda_{554}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; & y_{663} &= 0; & [y_{(13)}, y_2]_c &= \nu_2; & [[y_{5436}, y_3]_c, y_4]_c, y_3]_c &= 0; \\ y_3^2 &= \mu_3; & y_{443} &= 0; & [y_{(24)}, y_3]_c &= 0; & [y_{236}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_4 and A_2 .

10.12. **Type $\mathfrak{g}(8, 6)$** , $\zeta \in G'_3$. The Weyl groupoid has eight objects.

For $\begin{array}{ccccccc} \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{-1} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\zeta} & \circ \end{array}$, $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_7$, with defining relations

$$\begin{aligned} y_5^2 &= \mu_5; & y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= \lambda_{223}; \\ y_{665} &= 0; & y_{667} &= \lambda_{667}; & y_{332} &= \lambda_{332}; & y_{334} &= \lambda_{334}; \\ & & y_{443} &= \lambda_{443}; & y_{445} &= \lambda_{445}; & y_{776} &= \lambda_{776}; \\ y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; & [[y_{(36)}, y_5]_c, y_4]_c, y_5]_c &= [[y_{7654}, y_5]_c, y_6]_c, y_5]_c = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{56} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_5 and A_2 .

For $\begin{array}{c} \circ \\ -1 \\ \mid \\ \zeta \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated
 $\begin{array}{ccccccc} \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{-1} & \circ & \xrightarrow{\zeta} & \circ & \xrightarrow{-1} & \circ & \xrightarrow{\bar{\zeta}} & \circ \end{array}$

by y_i , $i \in \mathbb{I}_7$, with defining relations

$$\begin{aligned} y_4^2 &= \mu_4; & y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_5^2 &= \mu_5; & y_{223} &= \lambda_{223}; & y_{332} &= \lambda_{332}; & [y_{(35)}, y_4]_c &= [y_{347}, y_4]_c = 0; \\ y_7^2 &= \mu_7; & y_{334} &= 0; & y_{665} &= 0; & [y_{(46)}, y_5]_c &= [y_{657}, y_5]_c = 0 \\ & & y_{457} - q_{57}\bar{\zeta}[y_{47}, y_5]_c - q_{45}(1 - \zeta)y_5y_{47} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{47}, y_{57} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_6 and A_1 .

For $\begin{array}{c} \circ \\ \zeta \\ \mid \\ \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is gener-
 $\begin{array}{ccccccc} \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{-1} & \circ & \xrightarrow{\zeta} & \circ & \xrightarrow{-1} & \circ & \xrightarrow{\bar{\zeta}} & \circ \end{array}$

ated by y_i , $i \in \mathbb{I}_7$, with defining relations

$$y_4^2 = \mu_4; \quad y_{112} = \lambda_{112}; \quad y_{221} = \lambda_{221}; \quad y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1;$$

$$\begin{aligned} y_5^2 &= \mu_5; & y_{223} &= \lambda_{223}; & y_{332} &= \lambda_{332}; & [y_{(35)}, y_4]_c &= [[y_{65}, y_{657}]_c, y_5]_c = 0; \\ y_6^2 &= \mu_6; & y_{775} &= 0; & [y_{(46)}, y_5]_c &= \mu_5; & y_{334} &= [y_{457}, y_5]_c = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{57} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_6 and A_1 .

For
$$\begin{array}{c} \zeta \\ | \bar{\zeta} \\ \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \circ^{-1} \end{array}$$
 the algebra $\tilde{\mathcal{E}}_q$ is generated

by $y_i, i \in \mathbb{I}_7$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & y_{223} &= \lambda_{223}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{557} &= \lambda_{557}; & y_{332} &= \lambda_{332}; & y_{334} &= \lambda_{334}; & y_{443} &= \lambda_{443}; \\ y_{775} &= \lambda_{775}; & y_{445} &= \lambda_{445}; & y_{554} &= \lambda_{554}; & y_{556} &= 0; & y_6^2 &= \mu_2. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{56} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_6 and A_1 .

For
$$\begin{array}{c} \zeta \\ | \bar{\zeta} \\ \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \circ^{-1} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \end{array}$$
 the algebra $\tilde{\mathcal{E}}_q$ is generated

by $y_i, i \in \mathbb{I}_7$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_{112}; & y_{221} &= \lambda_{221}; & [y_{347}, y_4]_c &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_3^2 &= \mu_3; & y_{556} &= \lambda_{556}; & [y_{(24)}, y_3]_c &= 0; & [[y_{6547}, y_4]_c, y_5]_c, y_4]_c &= 0; \\ y_4^2 &= \mu_4; & y_{665} &= \lambda_{665}; & [y_{(35)}, y_4]_c &= 0; & y_{223} &= y_{774} = y_{554} = 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{47} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_6 and A_1 .

For
$$\begin{array}{c} \zeta \\ | \bar{\zeta} \\ \zeta \text{---} \bar{\zeta} \text{---} \circ^{-1} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \end{array}$$
 the algebra $\tilde{\mathcal{E}}_q$ is generated

by $y_i, i \in \mathbb{I}_7$, with defining relations

$$\begin{aligned} y_{112} &= y_{443} = 0; & y_{665} &= \lambda_{665}; & y_{445} &= \lambda_{445}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; & y_{447} &= \lambda_{447}; & y_{774} &= \lambda_{774}; & [y_{(13)}, y_2]_c &= 0; \\ y_3^2 &= \mu_3; & y_{554} &= \lambda_{554}; & y_{556} &= \lambda_{556}; & [y_{(24)}, y_3]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_q / \langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{q'}, q'$ with two components of types A_3 and A_4 .

For $\begin{array}{c} \zeta \\ \circ \\ \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated

$\begin{array}{ccccccc} \circ & \xrightarrow{\zeta} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ \\ \circ & \xrightarrow{\zeta} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ \end{array}$

by $y_i, i \in \mathbb{I}_7$, with defining relations

$$\begin{aligned} y_1^2 &= \mu_1; & y_{334} &= \lambda_{334}; & y_{443} &= \lambda_{443}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; & y_{445} &= \lambda_{445}; & y_{447} &= \lambda_{447}; & [y_{(13)}, y_2]_c &= 0; \\ y_{665} &= \lambda_{665} & y_{774} &= \lambda_{774}; & y_{554} &= \lambda_{554}; & y_{556} &= \lambda_{556}; & y_{332} &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{23} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_2 and D_5 .

For $\begin{array}{c} \zeta \\ \circ \\ \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated

$\begin{array}{ccccccc} \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ \\ \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ & \xrightarrow{\bar{\zeta}} & \circ \end{array}$

by $y_i, i \in \mathbb{I}_7$, with defining relations

$$\begin{aligned} y_{221} &= 0; & y_{223} &= \lambda_{223}; & y_{332} &= \lambda_{332}; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{556} &= \lambda_{556}; & y_{334} &= \lambda_{334}; & y_{443} &= \lambda_{443}; & y_{445} &= \lambda_{445}; \\ y_{665} &= \lambda_{665}; & y_{447} &= \lambda_{447}; & y_{774} &= \lambda_{774}; & y_{554} &= \lambda_{554}; & y_1^2 &= \mu_1. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{12} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_1 and E_6 .

11. SUPER MODULAR TYPE, CHARACTERISTIC 5

11.1. **Type** $\text{brj}(2; 5)$, $\zeta \in G'_5$. The Weyl groupoid has two objects.

For $\begin{array}{c} \zeta \\ \circ \\ \zeta^2 \\ \circ \end{array}$, $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_1, y_2 with defining relations

$$[y_{1112}, y_{112}]_c = \lambda_1; \quad y_2^2 = \lambda_2; \quad y_{11112} = 0; \quad [[y_{112}, y_{12}]_c, y_{12}]_c = 0,$$

with $\lambda_1 \neq 0$ only if $\mathbf{q} = \begin{pmatrix} \zeta & 1 \\ \zeta^2 & -1 \end{pmatrix}$. See `brj25.log`, resp. `brj252.log`, for the deformation of the first, resp. last, relation. If $\lambda_1 = 0$, then this algebra is nonzero by [A+, Lemma 5.16]. If $\lambda_1, \lambda_2 \neq 0$, then this algebra is nonzero by `brj25b.log`.

For $\begin{array}{c} -\zeta^3 \\ \circ \\ \zeta^3 \\ \circ \end{array}$, $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_1, y_2 with defining relations

$$y_2^2 = \lambda_2; \quad y_{111112} = 0; \quad [y_1, [y_{112}, y_{12}]_c]_c + q_{12} y_{112}^2 = 0.$$

This algebra is nonzero by [A+, Lemma 5.16].

11.2. **Type $\mathfrak{cl}(5; 5)$, $\zeta \in G'_5$.** The Weyl groupoid has seven objects.

For $\zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \underset{\circ}{\xrightarrow{-1}} \bar{\zeta} \underset{\circ}{\xrightarrow{\zeta}} \bar{\zeta}^2 \underset{\circ}{\xrightarrow{\zeta^2}}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{221} &= 0; & y_{223} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{554} &= \lambda_{554}; & y_{443} &= 0; & y_{4445} &= \lambda_{4445}; & [[y_{(14)}, y_3]_c, y_2]_c, y_3]_c &= 0; \\ y_3^2 &= \mu_3; & [[y_{5432}, y_4]_c, y_3]_c - q_{43}(\zeta^2 - \zeta)[[y_{5432}, y_3]_c, y_4]_c &= 0. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{34} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_3 and B_2 .

For $\zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \underset{\circ}{\xrightarrow{-1}} \zeta \underset{\circ}{\xrightarrow{-1}}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_4^2 &= \mu_4; & y_{221} &= 0; & y_{223} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_5^2 &= \mu_5; & y_{332} &= 0; & y_{334} &= 0; & [[y_{54}, y_{543}]_c, y_4]_c &= 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \zeta^2 \underset{\circ}{\xrightarrow{\bar{\zeta}^2}} \zeta \underset{\circ}{\xrightarrow{\bar{\zeta}}} \underset{\circ}{\xrightarrow{-1}}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i ,

$i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_5^2 &= \mu_5; & y_{112} &= 0; & y_{221} &= 0; & y_{223} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{445} &= 0; & y_{332} &= 0; & y_{334} &= \lambda_{334}; & y_{4443} &= \lambda_{4443}. \end{aligned}$$

Here, $\tilde{\mathcal{E}}_{\mathbf{q}}/\langle y_{45} \rangle \simeq \tilde{\mathcal{E}}_{\mathbf{q}'}$, \mathbf{q}' with two components of types A_1 and B_4 .

For $\begin{array}{c} \zeta^2 \\ \underset{\circ}{\mid} \\ \bar{\zeta}^2 \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,
 $\underset{\circ}{\xrightarrow{-1}} \zeta^2 \underset{\circ}{\xrightarrow{-1}} \bar{\zeta}^2 \underset{\circ}{\xrightarrow{\zeta^2}} \bar{\zeta}^2 \underset{\circ}{\xrightarrow{\zeta^2}}$

with defining relations

$$\begin{aligned} y_{332} &= 0; & y_1^2 &= \mu_1; & y_{334} &= 0; & y_{443} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_2^2 &= \mu_2; & y_{552} &= 0; & [y_{(13)}, y_2]_c &= 0; & [y_{125}, y_2]_c &= 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \zeta^2 \\ \underset{\circ}{\mid} \\ \bar{\zeta}^2 \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_5$,
 $\underset{\circ}{\xrightarrow{-1}} \bar{\zeta}^2 \underset{\circ}{\xrightarrow{\zeta^2}} \bar{\zeta}^2 \underset{\circ}{\xrightarrow{\zeta^2}} \bar{\zeta}^2 \underset{\circ}{\xrightarrow{\zeta^2}}$

with defining relations

$$y_1^2 = \mu_1; \quad y_{ij} = \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \quad y_{ii} = 0, \tilde{q}_{ij} \neq 1, i \neq 1.$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \circ \\ \zeta^2 \downarrow \zeta \\ \zeta^2 \quad \bar{\zeta}^2 \quad \circ \quad \zeta^2 \quad \circ \quad \bar{\zeta}^2 \quad \zeta^2 \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_5$,

with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{443} &= 0; & [y_{(13)}, y_2]_c &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ [y_{125}, y_2]_c &= 0; & y_2^2 &= \mu_2; & [y_{(24)}, y_3]_c &= 0; & [[y_{53}, y_{534}]_c, y_3]_c &= 0; \\ y_3^2 &= \mu_3; & y_5^2 &= \mu_5; & y_{235} - \frac{q_{35}}{\zeta^2 + \zeta} [y_{25}, y_3]_c - q_{23}(1 - \zeta)y_3y_{25} &= 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \zeta \\ \bar{\zeta} \downarrow \bar{\zeta} \\ \zeta^2 \quad \bar{\zeta}^2 \quad \zeta^2 \quad \bar{\zeta}^2 \quad \circ \quad \zeta^2 \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_5$,

with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{221} &= 0; & y_{223} &= 0; & y_{ij} &= \lambda_{ij}, i < j, \tilde{q}_{ij} = 1; \\ y_{554} &= 0; & y_{553} &= 0; & [y_{(24)}, y_3]_c &= 0; & [[[y_{1235}, y_3]_c, y_2]_c, y_3]_c &= 0; \\ y_3^2 &= \mu_3; & y_4^2 &= \mu_4; & y_{(35)} - q_{45}\zeta[y_{35}, y_4]_c - q_{34}(1 - \bar{\zeta})y_4y_{35} &= 0. \end{aligned}$$

This algebra is nonzero by [A+, Lemma 5.16].

12. UNIDENTIFIED TYPE

12.1. Type $\mathfrak{bgl}(4, \alpha)$, $q \neq \pm 1$. The Weyl groupoid has five objects. Two of them are obtained with $-q^{-1}$ instead of q ; hence we study the other three. We set $N = \text{ord } q$, $M = \text{ord } -q^{-1}$.

For $\begin{array}{c} q \quad q^{-1} \quad q \quad q^{-1} \quad \circ \quad -q \quad -q^{-1} \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_3^2 &= \lambda_3; & y_{13} &= 0; & y_{14} &= 0; & y_{24} &= 0; \\ y_{112} &= \lambda_1; & y_{221} &= \lambda_2; & y_{223} &= \lambda_4; & y_{443} &= \lambda_5. \end{aligned}$$

Here $\lambda_1\lambda_2 \neq 0$ or $\lambda_4 \neq 0$ only if $N = 3$. Also, $\lambda_1\lambda_4 = \lambda_2\lambda_4 = 0$. Similarly, $\lambda_5 \neq 0$ only if $M = 4$. In any case we can project onto a case of smaller rank and hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} q \quad q^{-1} \quad \circ \quad -1 \quad -1 \quad \circ \quad -q \quad -q^{-1} \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_4$,

with defining relations

$$\begin{aligned} y_2^2 &= \lambda_2; & y_{13} &= 0; & y_{14} &= 0; & y_{24} &= 0; \\ y_3^2 &= \lambda_3; & y_{23}^2 &= \lambda_4; & y_{112} &= \lambda_1; & y_{443} &= \lambda_5; \end{aligned}$$

$$[[y_{(14)}, y_2]_c, y_3]_c - q_{23} \frac{1+q}{1-q} [[y_{(14)}, y_3]_c, y_2]_c = 0.$$

Here, $\lambda_1 \lambda_5 = 0$ and λ_1 or λ_5 are nonzero only if $N = 4$. Thus, we can always project onto a case of smaller rank and hence $\tilde{\mathcal{E}}_q \neq 0$. See `bgla.log`, resp. `bgla1.log`, `bgla2.log` for the deformation of the longest relation when one, resp. two, resp. three, of the relations y_2^2, y_3^2, y_{23}^2 are deformed.

For $\begin{array}{c} q \\ \circ \xrightarrow{q^{-1}} \circ \xrightarrow{-1} \circ \xrightarrow{-1} \circ \xrightarrow{-q^{-1}} \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_1; & y_{13} &= 0; & y_{14} &= \lambda_5; & [y_{(13)}, y_2]_c &= 0; \\ y_2^2 &= \lambda_2; & y_{24}^2 &= \lambda_6; & y_3^2 &= \lambda_3; & y_4^2 &= \lambda_4; \end{aligned}$$

$$y_{(24)} + \frac{(1+q)q_{43}}{2} [y_{24}, y_3]_c - q_{23}(1+q^{-1})y_3y_{24} = 0.$$

Here $\lambda_1 \neq 0$ only if $N = 4$. Hence, if $\lambda_3 = 0$ we can project onto a case of smaller rank. If $\lambda_3 \neq 0$, then $\lambda_4 = \lambda_5 = \lambda_6 = 0$ and we can project onto a case of smaller rank. Assume $\lambda_1 = 0$. If $\lambda_5 \neq 0$, then we can project. Otherwise, $\lambda_2 = \lambda_4 = 0$ and we have that either $\lambda_6 = 0$ or $\lambda_3 = 0$. In any case, we can project. Hence $\tilde{\mathcal{E}}_q \neq 0$.

12.2. Type `ufo`(1), $\zeta \in G_4$. The Weyl groupoid has six objects.

For $\begin{array}{c} \zeta \quad \bar{\zeta} \\ \circ \xrightarrow{\quad} \circ \xrightarrow{\quad} \circ \xrightarrow{-1} \circ \xrightarrow{-1} \circ \xrightarrow{\bar{\zeta}} \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_3^2 &= \lambda_3; & y_{112} &= 0; & y_{221} &= 0; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1; \\ y_4^2 &= \lambda_4; & y_{223} &= \lambda_1; & y_{554} &= \lambda_2; & [[y_{(14)}, y_3]_c, y_2]_c, y_3]_c &= 0; \end{aligned}$$

$$y_{34}^2 = \lambda_5; \quad [[y_{(25)}, y_3]_c, y_4]_c - q_{34}(1+\zeta)[[y_{(25)}, y_4]_c, y_3]_c = 0.$$

Here $\tilde{\mathcal{E}}_q \neq 0$ as we can project onto a case of smaller rank by making $y_1 = 0$.

For $\begin{array}{c} \zeta \quad \bar{\zeta} \\ \circ \xrightarrow{\quad} \circ \xrightarrow{\quad} \circ \xrightarrow{\bar{\zeta}} \circ \xrightarrow{-1} \circ \xrightarrow{\bar{\zeta}} \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{221} &= 0; & y_4^2 &= \lambda_1; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1; \\ y_{223} &= 0; & y_{332} &= 0; & y_{334} &= \lambda_2; & y_{554} &= \lambda_3. \end{aligned}$$

In this case $\tilde{\mathcal{E}}_q \neq 0$ as we can project onto a case of smaller rank.

For $\begin{array}{c} \circ \\ \zeta \downarrow \\ \zeta \end{array} \begin{array}{c} -1 \\ \searrow \\ -1 \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\begin{array}{c} \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad \zeta \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$$

with defining relations

$$\begin{aligned} y_2^2 &= \lambda_2; & y_3^2 &= \lambda_3; & y_5^2 &= \lambda_5; & [y_{(13)}, y_2]_c &= 0; & y_{ij} &= 0, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_{112} &= \lambda_1; & y_{443} &= \lambda_4; & [y_{125}, y_2]_c &= 0; & [y_{(24)}, y_3]_c &= 0; \\ y_{35}^2 &= \lambda_6; & y_{235} + q_{35}(1 + \zeta)[y_{25}, y_3]_c - 2q_{23}y_3y_{25} &= 0. \end{aligned}$$

If $\lambda_1 = 0$, then we can project. If $\lambda_1\lambda_3 \neq 0$, then $\lambda_4 = 0$, and we can make $y_4 = 0$. If $\lambda_1\lambda_4 \neq 0$, then $\lambda_5 = 0$, and we can make $y_5 = 0$. Hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} \circ \\ -1 \downarrow \\ \zeta \end{array} \begin{array}{c} \zeta \\ \searrow \\ -1 \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\begin{array}{c} \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$$

with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{221} &= 0; & y_{223} &= \lambda_1; & y_{ij} &= 0, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_5^2 &= \lambda_2; & y_{35}^2 &= \lambda_5; & y_3^2 &= \lambda_3; & y_4^2 &= \lambda_4; & [[[y_{1235}, y_3]_c, y_2]_c, y_3]_c &= 0; \\ [y_{(24)}, y_3]_c &= 0; & y_{(35)} + \frac{q_{45}(1 + \zeta)}{2}[y_{35}, y_4]_c - q_{34}(1 - \zeta)y_4y_{35} &= 0. \end{aligned}$$

Hence $\tilde{\mathcal{E}}_q \neq 0$ as we can quotient by y_{34} , y_{45} .

For $\begin{array}{c} \zeta \\ \downarrow \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\begin{array}{c} -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$$

with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^2 &= \lambda_2; & [y_{(13)}, y_2]_c &= 0; & y_{ij} &= 0, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_{332} &= \lambda_3; & y_{334} &= 0; & y_{443} &= 0; & y_{552} &= \lambda_4; & [y_{125}, y_2]_c &= 0. \end{aligned}$$

We can project by making $y_4 = 0$, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} \zeta \\ \downarrow \\ \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_5$,

$$\begin{array}{c} -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$$

with defining relations

$$\begin{aligned} y_{221} &= \lambda_1; & y_{223} &= 0; & y_{225} &= 0; & y_{ij} &= 0, \quad i < j, \quad \tilde{q}_{ij} = 1; \\ y_1^2 &= \lambda_2; & y_{552} &= 0; & y_{332} &= 0; & y_{334} &= 0; & y_{443} &= 0. \end{aligned}$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

12.3. Type ufo(2), $\zeta \in G'_4$. The Weyl groupoid has seven objects.

For $\begin{array}{ccccccc} \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & -1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & & & & & & -1 \\ & & & & & & \circ \\ & & & & & & \bar{\zeta} \\ & & & & & & \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} = y_{221} = 0; \quad y_{334} = \lambda_1; \quad y_4^2 = \lambda_3; \quad y_{ij} = 0, i < j, \tilde{q}_{ij} = 1; \\ y_{332} = y_{223} = 0; \quad y_{665} = \lambda_2; \quad y_5^2 = \lambda_4; \quad [[y_{(25)}, y_4]_c, y_3]_c, y_4]_c = 0; \\ y_{45}^2 = \lambda_5; \quad [[y_{(36)}, y_4]_c, y_5]_c - q_{45}(1 + \zeta)[[y_{(36)}, y_5]_c, y_4]_c = 0. \end{aligned}$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{ccccccc} \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta & \bar{\zeta} & -1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & & & & & & \bar{\zeta} \\ & & & & & & \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{445} = \lambda_1; \quad y_{112} = 0; \quad y_{221} = 0; \quad y_{223} = 0; \quad y_{ij} = 0, i < j, \tilde{q}_{ij} = 1; \\ y_{665} = \lambda_2; \quad y_{332} = 0; \quad y_{334} = 0; \quad y_{443} = 0; \quad y_5^2 = \lambda_3 \end{aligned}$$

We can project by making $y_1 = y_2 = y_3 = 0$, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{ccccccc} & & & \zeta & & & \\ & & & \circ & & & \\ & & & | & & & \\ & & & \bar{\zeta} & & & \\ \zeta & \bar{\zeta} & -1 & \zeta & -1 & \bar{\zeta} & \zeta \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & & & & & & \bar{\zeta} \\ & & & & & & \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_{112} = \lambda_1; \quad y_{443} = \lambda_2; \quad y_2^2 = \lambda_4; \quad y_{ij} = 0, i < j, \tilde{q}_{ij} = 1; \quad y_{554} = 0 \\ y_{445} = 0; \quad y_{663} = \lambda_3; \quad y_3^2 = \lambda_5; \quad [y_{(13)}, y_2]_c = [y_{(24)}, y_3]_c = [y_{236}, y_3]_c = 0. \end{aligned}$$

Hence $\tilde{\mathcal{E}}_q \neq 0$ as we can project onto smaller rank.

For $\begin{array}{ccccccc} & & & \zeta & & & \\ & & & \circ & & & \\ & & & | & & & \\ & & & \bar{\zeta} & & & \\ -1 & \zeta & -1 & \bar{\zeta} & \zeta & \bar{\zeta} & \zeta \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & & & & & & \bar{\zeta} \\ & & & & & & \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_6$, with defining relations

$$\begin{aligned} y_1^2 = \lambda_1; \quad y_{332} = \lambda_2; \quad y_{336} = y_{663} = 0; \quad y_{ij} = 0, i < j, \tilde{q}_{ij} = 1; \\ y_2^2 = \lambda_3; \quad [y_{(13)}, y_2]_c = 0; \quad y_{334} = y_{443} = 0; \quad y_{445} = y_{554} = 0. \end{aligned}$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} \circ \\ \downarrow \begin{array}{l} -1 \\ \searrow \zeta \end{array} \\ \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_6$

\mathbb{I}_6 , with defining relations

$$\begin{aligned} y_4^2 &= \lambda_1; & y_5^2 &= \lambda_2; & [y_{(35)}, y_4]_c &= 0; & y_{223} &= 0; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1; \\ y_6^2 &= \lambda_3; & y_{46}^2 &= \lambda_4; & y_{332} &= 0; & y_{334} &= \lambda_5; & [[y_{2346}, y_4]_c, y_3]_c, y_4]_c &= 0; \\ y_{112} &= y_{221} = 0; & y_{(46)} &+ \frac{q_{56}(1+\zeta)}{2}[y_{46}, y_5]_c - q_{45}(1-\zeta)y_5y_{46} &= 0. \end{aligned}$$

We can project onto smaller rank by making $y_1 = y_2 = 0$, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} \zeta \\ \downarrow \bar{\zeta} \\ -1 \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_6$,

with defining relations

$$\begin{aligned} y_{223} &= y_{332} = 0; & y_{663} &= y_{336} = 0; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1; \\ y_{334} &= y_{443} = 0; & y_{445} &= y_{554} = 0; & y_1^2 &= \lambda_1; & y_{221} &= \lambda_2. \end{aligned}$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} \circ \\ \downarrow \begin{array}{l} -1 \\ \searrow -1 \end{array} \\ \zeta \quad \bar{\zeta} \quad \zeta \quad \bar{\zeta} \quad -1 \quad \zeta \quad -1 \quad \bar{\zeta} \quad \zeta \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_6$

\mathbb{I}_6 , with defining relations

$$\begin{aligned} y_3^2 &= \lambda_1; & y_4^2 &= \lambda_2; & y_{223} &= \lambda_3; & [y_{(24)}, y_3]_c &= 0; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1; \\ y_6^2 &= \lambda_4; & y_{46}^2 &= \lambda_5; & y_{554} &= \lambda_6; & [y_{(35)}, y_4]_c &= 0; & y_{112} &= y_{221} = 0; \\ [y_{236}, y_3]_c &= 0; & y_{346} &+ q_{46}(1+\zeta)[y_{36}, y_4]_c - 2q_{34}y_4y_{36} &= 0. \end{aligned}$$

We can project onto smaller rank by making $y_1 = y_2 = 0$, hence $\tilde{\mathcal{E}}_q \neq 0$.

12.4. Type ufo(3), $\zeta \in G'_3$. The Weyl groupoid has five objects.

For $\begin{array}{c} -1 \quad \bar{\zeta} \quad \zeta \quad -\bar{\zeta} \quad -\zeta \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by $y_i, i \in \mathbb{I}_3$, with defining relations

$$y_1^2 = \lambda_1; \quad y_2^3 = \lambda_2; \quad y_{13} = 0; \quad y_{221} = 0; \quad y_{332} = \lambda_3,$$

with $\lambda_1\lambda_3 = \lambda_2\lambda_3 = 0$. Hence $\tilde{\mathcal{E}}_q$ projects onto a nonzero rank-2 case.

For $\begin{array}{c} -1 & \zeta & -1 & -\bar{\zeta} & -\zeta \\ \circ & \xrightarrow{\quad} & \circ & \xrightarrow{\quad} & \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = 0; \quad y_{332} = 0; \quad y_1^2 = \lambda_1; \quad y_2^2 = \lambda_2.$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \zeta & -1 & -1 & -\bar{\zeta} & -\zeta \\ \circ & \xrightarrow{\quad} & \circ & \xrightarrow{\quad} & \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_1^3 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [y_{112}, y_{12}]_c = \lambda_3; \quad [[y_{12}, y_{(13)}]_c, y_2]_c = y_{13} = y_{332} = 0.$$

Here $\tilde{\mathcal{E}}_{\mathfrak{q}}$ projects onto a rank 2 case by making $y_3 = 0$.

For $\begin{array}{c} \zeta & -\bar{\zeta} & -\zeta & -\bar{\zeta} & -\zeta \\ \circ & \xrightarrow{\quad} & \circ & \xrightarrow{\quad} & \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_{13} = 0; \quad y_{221} = \lambda_2; \quad y_1^3 = \lambda_1; \quad y_{223} = 0; \quad y_{332} = 0.$$

This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} & -1 & \\ & \circ & \\ \zeta & \nearrow & \searrow -\zeta \\ \circ & \xrightarrow{-1} & \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with

defining relations

$$y_{113} = 0; \quad y_1^3 = \lambda_1; \quad [y_{112}, y_{12}]_c = \lambda_3; \quad y_2^2 = \lambda_2; \quad y_3^2 = \lambda_4 \\ y_{(13)} - \frac{q_{23}\zeta}{1-\bar{\zeta}}[y_{13}, y_2]_c + q_{12}\bar{\zeta}y_2y_{13} = 0,$$

with $\lambda_2\lambda_4 = 0$. Hence the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ projects onto a rank-2 nonzero algebra by setting $y_2 = 0$ or $y_3 = 0$ according to whether $\lambda_4 = 0$ or $\lambda_2 = 0$.

12.5. Type $\mathbf{ufo}(4)$, $\zeta \in G'_3$. The Weyl groupoid has nine objects.

For $\begin{array}{c} -1 & -1 & -1 & \zeta & -1 \\ \circ & \xrightarrow{\quad} & \circ & \xrightarrow{\quad} & \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_1^2 = \lambda_1; \quad y_2^2 = \lambda_2; \quad y_3^2 = \lambda_3; \quad y_{23}^3 = \lambda_6; \\ y_{12}^2 = \lambda_4; \quad y_{13} = \lambda_5; \quad [y_{123}, [y_{123}, y_{23}]_c]_c = \lambda_7$$

with $\lambda_2\lambda_3 = \lambda_7\lambda_5 = 0$. The deformation of y_{23}^3 follows from **ufo4a1.log** and of the longest relation by **ufo4a2.log**.

Assume $\lambda_7 = 0$. Then we can project onto a case of smaller rank unless $\lambda_1\lambda_2\lambda_4\lambda_6 \neq 0$ or $\lambda_1\lambda_3\lambda_6 \neq 0$, which gives $\mathfrak{q} = \begin{pmatrix} -1 & \pm 1 & \pm 1 \\ \mp 1 & -1 & -\zeta \\ \pm 1 & -1 & -1 \end{pmatrix}$ or $\mathfrak{q} = \begin{pmatrix} -1 & \pm 1 & \pm 1 \\ \mp 1 & -1 & -1 \\ \pm 1 & -\zeta & -1 \end{pmatrix}$. In any case, $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$, see **ufo4a.log**.

Now $\lambda_7 \neq 0$ only if $\mathfrak{q} = \begin{pmatrix} -1 & a & a^{-1} \\ -a^{-1} & -1 & c \\ a & c^{-1}\zeta & -1 \end{pmatrix}$, with $c^3 = -a^2$. In any case, $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$, see `ufo4a7.log`.

For $\begin{smallmatrix} -1 & -1 \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} -1 & -\bar{\zeta} \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} \bar{\zeta} \\ \circ \end{smallmatrix}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^3 &= \lambda_3 & [y_{332}, y_{32}]_c &= \lambda_4; \\ [y_{3321}, y_{321}]_c &= \lambda_5; & y_{13} &= 0; & y_{12}^2 &= \lambda_6; & [[y_{32}, y_{321}]_c, y_2]_c &= 0. \end{aligned}$$

We can project onto a case of rank two unless $\mathfrak{q} = \begin{pmatrix} -1 & \pm 1 & 1 \\ \mp 1 & -1 & \zeta^{-1} \\ 1 & -1 & \zeta^{-1} \end{pmatrix}$, when all scalars can be nonzero. In this case, $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by `ufo4b.log`.

For $\begin{smallmatrix} -1 & -1 \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} \zeta & \bar{\zeta} \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^3 &= \lambda_2; & y_3^2 &= \lambda_3; & [y_{221}, y_{21}]_c &= \lambda_5; \\ y_{223} &= 0; & y_{13} &= \lambda_4; & [y_{12}, y_{(13)}]_c &= 0, \end{aligned}$$

with $\lambda_1\lambda_2 = 0 = \lambda_2\lambda_3 = \lambda_2\lambda_5 = \lambda_2\lambda_4$. Hence, if $\lambda_2 \neq 0$, then $\tilde{\mathcal{E}}_{\mathfrak{q}}$ projects onto $\mathbb{k}\langle y_2 | y_2^3 = \lambda_2 \rangle$. If $\lambda_1 \neq 0$, then $\lambda_5 = 0$ and thus $\tilde{\mathcal{E}}_{\mathfrak{q}}$ projects onto the linking of two rank-one algebras, by making $y_2 = 0$. A similar situation holds when $\lambda_3 \neq 0$ and $\lambda_4 = 0$. On the other hand, $\lambda_3\lambda_4 \neq 0$ only when $\mathfrak{q} = \begin{pmatrix} -1 & \mp 1 & -1 \\ \pm 1 & \zeta & \pm 1 \\ \mp 1 & \pm \zeta^2 & -1 \end{pmatrix}$ and $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by `ufo4c.log`.

For $\begin{smallmatrix} -1 & \zeta \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} -1 & -\zeta \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_1^2 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [[y_{32}, y_{321}]_c, y_2]_c = \lambda_5; \quad y_3^2 = \lambda_3; \quad y_{13} = \lambda_4.$$

Here $\lambda_5 \neq 0$ only if $\mathfrak{q} = \begin{pmatrix} -1 & -c^2\zeta & -c^{-3} \\ -c^{-2} & -1 & c \\ -c^3 & -c^{-1}\zeta & -1 \end{pmatrix}$, some $c \in \mathbb{k}^\times$. Otherwise $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by making $y_{23} = 0$. When $\lambda_5 \neq 0$ is the only nonzero scalar then $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by `ufo4d.log`. Then $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by Lemma 3.3.

For $\begin{smallmatrix} -1 & \zeta \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{smallmatrix} \xrightarrow{\quad} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$, the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$y_1^2 = \lambda_1; \quad y_{2221} = \lambda_2; \quad y_3^2 = \lambda_3; \quad y_{13} = \lambda_4 \quad y_{223} = 0.$$

Here $\lambda_2 \neq 0$ only if $\mathfrak{q} = \begin{pmatrix} -1 & -\zeta & * \\ -1 & -\bar{\zeta} & * \\ * & * & -1 \end{pmatrix}$. If $\lambda_2 = 0$, $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$, by making $y_2 = 0$.

A similar argument shows that $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ when $\lambda_2 \neq 0$: $\tilde{\mathcal{E}}_{\mathfrak{q}}$ arises as the linking of a rank-two case (generated by y_1, y_2) and a rank-one case.

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{-\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^3 &= \lambda_2; & y_3^2 &= \lambda_3; & y_{13} &= \lambda_4; & [y_{223}, y_{23}]_c &= \lambda_5; \\ [y_1, y_{223}]_c + q_{23}[y_{123}, y_2]_c - q_{12}y_2y_{123} &= 0, \end{aligned}$$

with $\lambda_2\lambda_4 = 0$. We may project onto a case of smaller rank if $\lambda_4 = \lambda_1 = 0$ or $\lambda_4 = \lambda_3 = \lambda_5 = 0$, so $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$ in these cases. The same holds if $\lambda_2 = \lambda_5 = 0$. Now, if $\lambda_1\lambda_4 \neq 0$, then $\lambda_5 = 0$; hence $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$.

If $\lambda_2\lambda_5 \neq 0$ or $\lambda_2\lambda_3 \neq 0$ and besides $\lambda_1 \neq 0$, then $\mathbf{q} = \begin{pmatrix} -1 & \zeta^2 & \pm 1 \\ 1 & \zeta^2 & 1 \\ \pm 1 & \zeta^2 & -1 \end{pmatrix}$. In any case $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$: on the one hand this holds if $\lambda_5 \neq 0$ is the unique nonzero scalar, by making $y_1 = 0$, and on the other when $\lambda_4 = \lambda_5 = 0$ by $[A+, \text{Lemma 5.16}]$; hence $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$ using Lemma 3.3.

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{-\zeta} \begin{array}{c} -1 \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^3 &= \lambda_2; & y_3^2 &= \lambda_3; \\ y_{221} &= 0; & y_{13} &= \lambda_4; & [y_{223}, y_{23}]_c &= \lambda_5. \end{aligned}$$

Here $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$ by making $y_{12} = 0$.

For $\begin{array}{c} -1 \\ \circ \end{array} \xrightarrow{-1} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{-\zeta} \begin{array}{c} -1 \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^3 &= \lambda_2; & y_3^2 &= \lambda_3; & [y_{221}, y_{21}]_c &= \lambda_4; & y_{13}^2 &= \lambda_5; & y_{223} &= 0; \\ y_{(13)} + \frac{q_{23}(1-\zeta)}{2}[y_{13}, y_2]_c - q_{12}(1-\bar{\zeta})y_2y_{13} &= 0. \end{aligned}$$

We can project onto a case of smaller rank and thus show that $\tilde{\mathcal{E}}_{\mathbf{q}}$ is nonzero whenever $\mathbf{q} \neq \begin{pmatrix} -1 & \zeta & \pm 1 \\ -1 & \zeta & 1 \\ \mp 1 & \zeta^2 & -1 \end{pmatrix}$. Still, $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$, by `uf04h.log`.

For $\begin{array}{c} \zeta \\ \circ \end{array} \xrightarrow{-\bar{\zeta}} \begin{array}{c} \bar{\zeta} \\ \circ \end{array} \xrightarrow{-\bar{\zeta}} \begin{array}{c} -1 \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_{\mathbf{q}}$ is generated by y_i , $i \in \mathbb{I}_3$, with defining relations

$$\begin{aligned} y_{113} &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^3 &= \lambda_3 & y_{112} &= 0; & y_{332} &= 0; \\ y_{(13)} + q_{23}(1-\bar{\zeta})[y_{13}, y_2]_c - q_{12}(1-\bar{\zeta})y_2y_{13} &= 0. \end{aligned}$$

Here $\lambda_1 \neq 0$ only if $\mathbf{q} = \begin{pmatrix} -\zeta & * & \zeta \\ * & -1 & * \\ -\zeta & * & \zeta \end{pmatrix}$ and $\lambda_1\lambda_2 = 0$. Hence $\tilde{\mathcal{E}}_{\mathbf{q}} \neq 0$, by projecting onto a rank 2 case.

12.6. **Type** $\mathfrak{uf}\mathfrak{o}(5)$, $\zeta \in G'_3$. The Weyl groupoid has six objects.

For $\begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} \zeta \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{221} &= 0; & y_4^3 &= \lambda_1; \\ y_{223} &= 0; & y_{332} &= 0; & y_{334} &= \lambda_2; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1. \end{aligned}$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -1 & -1 \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} \zeta \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{112} &= 0; & y_{221} &= 0; & [y_{443}, y_{43}]_c &= 0; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1; \\ y_{223} &= 0; & y_3^2 &= \lambda_1; & y_4^3 &= \lambda_2; & [[y_{43}, y_{432}]_c, y_3]_c &= 0. \end{aligned}$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} \zeta & \bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -1 & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -\zeta \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$y_{112} = \lambda_1; \quad y_{443} = 0; \quad y_{ij} = 0, i < j, \tilde{q}_{ij} = 1; \quad y_2^3 = \lambda_2; \quad y_3^2 = \lambda_3.$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -\zeta & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -1 & -\bar{\zeta} \\ \circ & \circ \end{array} \xrightarrow{\quad} \begin{array}{c} -\zeta \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$y_{112} = y_{221} = y_{223} = y_{443} = 0; \quad y_{ij} = 0, i < j, \tilde{q}_{ij} = 1; \quad y_3^2 = \lambda_1.$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} -1 \\ \circ \end{array} \begin{array}{c} -\zeta \\ \circ \end{array} \begin{array}{c} -\bar{\zeta} \\ \circ \end{array} \begin{array}{c} -1 \\ \circ \end{array} \begin{array}{c} \zeta \\ \circ \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} [y_{334}, y_{34}]_c &= \lambda_1; & y_{112} &= 0; & y_2^2 &= \lambda_2; & y_{ij} &= 0, i < j, \tilde{q}_{ij} = 1; \\ [y_{124}, y_2]_c &= 0; & y_{332} &= 0; & y_3^3 &= \lambda_3; & y_4^2 &= \lambda_4; \end{aligned}$$

$$y_{(24)} - 2q_{34}\zeta[y_{24}, y_3]_c - 2q_{23}y_3y_{24} = 0.$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

For $\begin{array}{c} \circ \\ \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \\ \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \end{array}$, the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with

defining relations

$$\begin{aligned} y_2^2 &= \lambda_1; & y_3^2 &= \lambda_2; & y_4^2 &= \lambda_3; & y_{ij} &= 0, \ i < j, \ \tilde{q}_{ij} = 1; \\ y_{112} &= 0; & [y_{124}, y_2]_c &= 0; & y_{(24)} + q_{34}\bar{\zeta}[y_{24}, y_3]_c + q_{23}\bar{\zeta}y_3y_{24} &= 0. \end{aligned}$$

We can project onto a case of smaller rank, hence $\tilde{\mathcal{E}}_q \neq 0$.

12.7. Type $\mathfrak{uf}\mathfrak{o}(6)$, $\zeta \in G'_4$. The Weyl groupoid has eight objects.

For $\begin{array}{c} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^4 &= \lambda_7; & y_{332} &= \lambda_3; \\ y_{13} &= \lambda_5; & y_{443} &= 0; & y_{14} &= y_{24} = 0; & [y_{(13)}, y_2]_c &= \lambda_4; \\ & & & & & & [[y_{(24)}, y_3]_c, y_3]_c &= \lambda_6. \end{aligned}$$

Here $\lambda_1\lambda_3 = \lambda_1\lambda_4 = \lambda_1\lambda_5 = 0 = \lambda_1\lambda_6$. Hence we can project onto a case of smaller rank and thus $\tilde{\mathcal{E}}_q \neq 0$. See `ufo6a.log` for the deformation of the last relation.

For $\begin{array}{c} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^2 &= \lambda_3; & [y_{(13)}, y_2]_c &= \lambda_4; \\ y_{443} &= \lambda_5; & y_{13} &= \lambda_6; & y_{14} &= y_{24} = 0; & [[y_{23}, [y_{23}, y_{(24)}]_c]_c, y_3]_c &= 0. \end{aligned}$$

If $\lambda_5 = 0$, then $\tilde{\mathcal{E}}_q \neq 0$ by making $y_4 = 0$. Similarly if λ_5 is the unique nonzero scalar. The general case now follows from Lemma 3.3.

For $\begin{array}{c} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_{221} &= \lambda_2; & y_3^2 &= \lambda_3; & y_{223} &= \lambda_4; \\ y_{443} &= \lambda_5; & y_{13} &= \lambda_6; & y_{14} &= y_{24} = 0. \end{aligned}$$

If $\lambda_5\lambda_4 \neq 0$, then $\lambda_1 = \lambda_2 = \lambda_6 = 0$ and hence we can project by making $y_1 = 0$, so $\tilde{\mathcal{E}}_q \neq 0$. If $\lambda_5 \neq 0$ and $\lambda_4 = 0$, then $\tilde{\mathcal{E}}_q \neq 0$ by making $y_{23} = 0$. If $\lambda_5 = 0$, then $\tilde{\mathcal{E}}_q \neq 0$ by making $y_4 = 0$.

For $\begin{array}{c} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \text{---} \zeta \text{---} \bar{\zeta} \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_{221} &= \lambda_2; & y_3^2 &= \lambda_3; & y_{13} &= \lambda_4; \\ y_{443} &= \lambda_5; & y_{2223} &= 0; & y_{14} &= 0; & y_{24} &= 0; \end{aligned}$$

$$[y_2, [y_{(24)}, y_3]_c]_c = \frac{q_{23}q_{43}}{1+\zeta} [y_{23}, y_{(24)}]_c + (\zeta - 1)q_{23}q_{43}y_{(24)}y_{23}.$$

If $\lambda_5 = 0$, then $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by making $y_4 = 0$. Similarly if λ_5 is the unique nonzero scalar. The general case now follows from Lemma 3.3.

For $\begin{array}{c} -1 & \zeta & -1 & -1 & -1 & \zeta & \bar{\zeta} \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^2 &= \lambda_3; & y_{443} &= \lambda_4; \\ y_{23}^2 &= \lambda_5; & y_{13} &= \lambda_6 & y_{14} &= y_{24} = 0; & [[y_{12}, y_{(13)}]_c, y_2]_c &= \lambda_7. \end{aligned}$$

If $\lambda_4 = 0$, then $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by making $y_4 = 0$. Similarly if λ_4 is the unique nonzero scalar. The general case now follows from Lemma 3.3.

For $\begin{array}{c} \zeta \\ \bar{\zeta} \mid \searrow \bar{\zeta} \\ \bar{\zeta} \text{---} \zeta \text{---} -1 \text{---} -1 \text{---} -1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_{112} &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^2 &= \lambda_3; & y_{443} &= \lambda_4; & y_{442} &= \lambda_5; \\ y_{23}^2 &= \lambda_6; & y_{14} &= \lambda_7; & [y_{124}, y_2]_c &= \lambda_8; & y_{13} &= 0; \end{aligned}$$

$$y_{(24)} - q_{34}\zeta[y_{24}, y_3]_c - q_{23}(1 + \zeta)y_3y_{24} = 0.$$

We analyze the cases in which we cannot project onto a case of smaller rank, notice that, for instance, $\lambda_1\lambda_5 \neq 0$ or $\lambda_1\lambda_5 \neq 0$, imply that $\lambda_3 = \lambda_4 = \lambda_6 = 0$ and we can make $y_3 = 0$. Similarly if $\lambda_7 \neq 0$ or $\lambda_8 \neq 0$.

We are left with the case $\lambda_1\lambda_4 \neq 0$, in which $\mathfrak{q} = \begin{pmatrix} \bar{\zeta} & -1 & 1 & \pm 1 \\ \bar{\zeta} & -1 & -1 & \bar{\zeta} \\ 1 & 1 & -1 & \zeta \\ \pm 1 & 1 & -1 & \zeta \end{pmatrix}$, and we could also have $\lambda_2\lambda_3\lambda_6 \neq 0$; on the other hand, $\lambda_5 = \lambda_7 = \lambda_8 = 0$. In any case, $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ by `ufo6f.g`.

For $\begin{array}{c} -1 \\ \circ \mid \searrow \bar{\zeta} \\ -1 \text{---} \bar{\zeta} \text{---} \zeta \text{---} -1 \text{---} -1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_{\mathfrak{q}}$ is generated by y_i , $i \in \mathbb{I}_4$, with defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_{13} &= \lambda_2 & y_3^2 &= \lambda_3; & y_4^2 &= \lambda_4; \\ y_{221} &= \lambda_5; & y_{224} &= \lambda_6 & y_{14} &= \lambda_7; & y_{2223} &= 0; \end{aligned}$$

$$y_{(24)} - q_{34}\zeta[y_{24}, y_3]_c - q_{23}(1 + \zeta)y_3y_{24} = 0.$$

Here $\tilde{\mathcal{E}}_{\mathfrak{q}} \neq 0$ as we can quotient by y_{23}, y_{34} .

For $\begin{array}{c} \circ \\ \bar{\zeta} \mid \\ \circ \end{array} \begin{array}{c} -1 \\ \diagdown \bar{\zeta} \\ \circ \end{array}$ the algebra $\tilde{\mathcal{E}}_q$ is generated by y_i , $i \in \mathbb{I}_4$, with

$$\begin{array}{c} \circ \\ \bar{\zeta} \mid \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \circ \\ -1 \\ \circ \end{array} \xrightarrow{-1} \begin{array}{c} \circ \\ -1 \\ \circ \end{array}$$

defining relations

$$\begin{aligned} y_1^2 &= \lambda_1; & y_2^2 &= \lambda_2; & y_3^2 &= \lambda_3; & y_4^2 &= \lambda_4; & [y_{124}, y_2]_c &= \lambda_5; \\ y_{14} &= \lambda_6; & y_{23}^2 &= \lambda_7; & y_{13} &= \lambda_8; & [[y_{12}, y_{(13)}]_c, y_2]_c &= \lambda_9; \\ y_{(24)} &- q_{34}\zeta[y_{24}, y_3]_c - q_{23}(1 + \zeta)y_3y_{24} &= 0. \end{aligned}$$

If $\lambda_5 = 0$, then $\tilde{\mathcal{E}}_q \neq 0$ since we can quotient by y_{24} , y_{34} . Similarly if λ_4 is the unique nonzero scalar. The general case now follows from Lemma 3.3.

12.8. Type $\mathbf{ufo}(7)$, $\zeta \in G'_{12}$. The Weyl groupoid has five objects but two of them are obtained with ζ^5 instead of ζ . Hence we focus in the remaining three cases.

For $\begin{array}{c} \circ \\ -\bar{\zeta}^2 \\ \circ \end{array} \xrightarrow{-\zeta^3} \begin{array}{c} \circ \\ -\zeta^2 \\ \circ \end{array}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^3 = \lambda_2; \quad [y_1, y_{122}]_c - \frac{\zeta^{10}(1 + \zeta)q_{12}}{1 + \zeta^3}y_{12}^2 = 0,$$

with $\lambda_1\lambda_2 = 0$. This is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \circ \\ -\bar{\zeta}^2 \\ \circ \end{array} \xrightarrow{\bar{\zeta}} \begin{array}{c} \circ \\ -1 \\ \circ \end{array}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [[y_{112}, y_{12}]_c, y_{12}]_c = 0,$$

with $\lambda_1\lambda_2 = 0$. The last relation is undeformed in any case, see $\mathbf{ufo}(7) \cdot \log$. This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \circ \\ -\zeta^3 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \circ \\ -1 \\ \circ \end{array}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^4 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [y_{112}, y_{12}]_c = 0,$$

with $\lambda_1\lambda_2 = 0$. This is nonzero by [A+, Lemma 5.16].

12.9. Type $\mathbf{ufo}(8)$, $\zeta \in G'_{12}$. The Weyl groupoid has three objects.

For $\begin{array}{c} \circ \\ -\zeta^2 \\ \circ \end{array} \xrightarrow{\zeta} \begin{array}{c} \circ \\ -\zeta^2 \\ \circ \end{array}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^3 = \lambda_2; \quad [y_1, y_{122}]_c - (1 + \zeta + \zeta^2)\zeta^4q_{12}y_{12}^2 = 0,$$

with $\lambda_1\lambda_2 = 0$. This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{array}{c} \circ \\ -\zeta^2 \\ \circ \end{array} \xrightarrow{\zeta^3} \begin{array}{c} \circ \\ -1 \\ \circ \end{array}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [[y_{112}, y_{12}]_c, y_{12}]_c = 0.$$

with $\lambda_1\lambda_2 = 0$. The last relation is undeformed in any case, see `ufo8.log`. This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{smallmatrix} \bar{\zeta} \\ \circ \end{smallmatrix} \xrightarrow{-\zeta^3} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_{11112} = 0; \quad y_2^2 = \lambda_2; \quad [y_{112}, y_{12}]_c = 0.$$

This algebra is nonzero by [A+, Lemma 5.16].

12.10. **Type** `ufo(9)`, $\zeta \in G'_{24}$. The Weyl groupoid has four objects.

For $\begin{smallmatrix} \zeta^6 \\ \circ \end{smallmatrix} \xrightarrow{-\bar{\zeta}} \begin{smallmatrix} -\bar{\zeta}^4 \\ \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^4 = \lambda_1; \quad y_2^3 = \lambda_2; \quad [y_1, y_{122}]_c - \frac{1+\zeta^7}{1+\zeta}\zeta^{10}q_{12}y_{12}^2 = 0,$$

with $\lambda_1\lambda_2 = 0$. This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{smallmatrix} \zeta^6 \\ \circ \end{smallmatrix} \xrightarrow{\zeta} \begin{smallmatrix} \bar{\zeta} \\ \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^4 = \lambda_1; \quad y_{221} = 0; \\ [[y_{112}, y_{12}]_c, y_{12}]_c = -\lambda_1 q_{21}^{-1}(2\zeta + \zeta^8 + 2\zeta^{17} - \zeta^{22})y_2^3.$$

See `ufo9b.log` and `ufo9b2.log`. This algebra is nonzero, as it projects onto $\mathbb{K}\langle y_1 | y_1^4 = \lambda_1 \rangle$.

For $\begin{smallmatrix} -\bar{\zeta}^4 \\ \circ \end{smallmatrix} \xrightarrow{\zeta^5} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [y_{112}, [[y_{112}, y_{12}]_c, y_{12}]_c - \alpha[y_{112}, y_{12}]_c^2 = 0,$$

for $\alpha = \frac{1+\zeta+\zeta^6+2\zeta^7+\zeta^{17}}{1+\zeta^4+\zeta^6+\zeta^{11}}\zeta^9q_{12}$. Here, $\lambda_1\lambda_2 = 0$. The last relation remains undeformed, see `ufo9c.log`. This algebra is nonzero by [A+, Lemma 5.16].

For $\begin{smallmatrix} \zeta \\ \circ \end{smallmatrix} \xrightarrow{\bar{\zeta}^5} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_2^2 = \lambda_1; \quad y_{111112} = 0; \quad [y_{112}, y_{12}]_c = 0$$

This algebra is nonzero by [A+, Lemma 5.16]. The last relation remains undeformed by direct computation.

12.11. **Type** `ufo(10)`, $\zeta \in G'_{20}$. The Weyl groupoid has four objects but two of them are obtained with $-\zeta$ instead of ζ . Hence we focus in just two cases.

For $\begin{smallmatrix} \zeta \\ \circ \end{smallmatrix} \xrightarrow{\bar{\zeta}^3} \begin{smallmatrix} -1 \\ \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_2^2 = \lambda_1; \quad y_{11112} = 0; \quad [[[y_{112}, y_{12}]_c, y_{12}]_c, y_{12}]_c = 0.$$

This algebra is nonzero by [A+, Lemma 5.16]. The last relation remains undeformed by `ufo10a.log`.

For $\begin{smallmatrix} -\zeta^2 & \zeta^3 \\ \circ & \circ \end{smallmatrix}^{-1}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^5 = \lambda_1; \quad y_2^2 = \lambda_2 \quad [y_1, [y_{112}, y_{12}]_c]_c + \frac{1 - \zeta^{17}}{1 - \zeta^2} q_{12} y_{112}^2 = 0,$$

with $\lambda_1 \lambda_2 = 0$. The last relation remains undeformed, see `ufo10c.log`, and thus this algebra is nonzero by [A+, Lemma 5.16].

12.12. **Type** `ufo(11)`, $\zeta \in G'_{15}$. The Weyl groupoid has four objects.

For $\begin{smallmatrix} -\zeta & -\zeta^3 & \zeta^5 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_2^3 = \lambda_2; \quad y_{11112} = 0; \quad [[y_{112}, y_{12}]_c, y_{12}]_c = 0; \quad [y_1, y_{122}]_c + \alpha y_{12}^2 = 0,$$

for $\alpha = \frac{1+\zeta^{13}}{1+\zeta^{12}} \zeta^{10} q_{12}$. This algebra is nonzero by [A+, Lemma 5.16]. The third relation is undeformed by `ufo11a1.log`. We recall that the fourth one is primitive in $T(V)$.

For $\begin{smallmatrix} \zeta^3 & -\zeta^4 & -\zeta^4 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$[y_1, [y_{112}, y_{12}]_c]_c = \frac{1 - \zeta^2}{1 + \zeta^7} \zeta^9 q_{12} y_{112}^2; \quad y_1^5 = \lambda_1; \quad y_{221} = 0;$$

$$[[[y_{112}, y_{12}]_c, y_{12}]_c, y_{12}]_c = \lambda_1 q_{21} (\zeta + \zeta^4 + 3\zeta^8 + 4\zeta^{13} - 3\zeta^{14}) y_2^4,$$

See `ufo11b.log` for the rule of deformation of the bottom quantum Serre relation. This algebra is nonzero as it projects onto $\mathbb{k}\langle y_1 | y_1^5 = \lambda_1 \rangle$.

For $\begin{smallmatrix} \zeta^5 & -\zeta^2 & -1 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^3 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [[[y_{112}, y_{12}]_c, y_{12}]_c, y_{12}]_c = 0,$$

with $\lambda_1 \lambda_2 = 0$. The third relation is undeformed by `ufo11c.log` and thus this algebra is nonzero by [A+, Lemma 5.16].

For $\begin{smallmatrix} \zeta^3 & -\zeta^2 & -1 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_1^5 = \lambda_1; \quad y_2^2 = \lambda_2; \quad [y_{112}, y_{12}]_c = 0; \quad [[y_{1112}, y_{112}]_c, y_{112}]_c = 0,$$

with $\lambda_1 \lambda_2 = 0$. The third relation is undeformed by direct computation. As for the fourth, see `ufo11d.log`. Then this algebra is nonzero by [A+, Lemma 5.16].

12.13. **Type** `ufo(12)`, $\zeta \in G'_7$. The Weyl groupoid has two objects.

For $\begin{smallmatrix} -\zeta & -\zeta^3 & -1 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_{11112} = 0; \quad y_2^2 = \lambda_2; \quad [y_{112}, [[y_{112}, y_{12}]_c, y_{12}]_c]_c - \alpha [y_{112}, y_{12}]_c^2 = 0,$$

for $\alpha = -q_{12} \frac{-\zeta + 3\zeta^2 + 3\zeta^3 - \zeta^4 + 3\zeta^6}{-2\zeta + 2\zeta^3 - \zeta^5 + \zeta^6}$. By `ufo12a.log`, the last relation is undeformed. The algebra is nonzero as it projects onto $\mathbb{k}\langle y_2^2 | y_2^2 = \lambda_2 \rangle$.

For $\begin{smallmatrix} -\zeta^2 & -\zeta^3 & -1 \\ \circ & \circ & \circ \end{smallmatrix}$, $\tilde{\mathcal{E}}_q$ is generated by y_1, y_2 with defining relations

$$y_{1111112} = 0; \quad y_2^2 = \lambda_2; \quad [y_1, [y_{112}, y_{12}]_c]_c + q_{12} \frac{1 + \zeta^4}{1 - \zeta^2} y_{112}^2 = 0.$$

The last relation is undeformed by direct computation. The algebra is nonzero as it projects onto $\mathbb{k}\langle y_2^2 | y_2^2 = \lambda_2 \rangle$.

REFERENCES

- [AA] N. ANDRUSKIEWITSCH, I. ANGIONO. *Finite dimensional Nichols algebras of diagonal type*. In progress.
- [A+] N. ANDRUSKIEWITSCH, I. ANGIONO, A. GARCÍA IGLESIAS, A. MASUOKA, C. VAY. *Lifting via cocycle deformation*. J. Pure Appl. Alg. **218** (4), 684–703 (2014).
- [AAG] N. ANDRUSKIEWITSCH, I. ANGIONO, A. GARCÍA IGLESIAS. *Liftings of Nichols algebras of diagonal type I. Cartan type A*. Int. Math. Res. Not. IMRN, in press. Available at [arXiv:1509.01622](https://arxiv.org/abs/1509.01622).
- [AG] N. ANDRUSKIEWITSCH, A. GARCÍA IGLESIAS. *Twisting Hopf algebras from cocycle deformations*. Submitted, available at [arXiv:1512.01696](https://arxiv.org/abs/1512.01696).
- [AS1] N. ANDRUSKIEWITSCH, H.-J. SCHNEIDER. *Isomorphism classes and automorphisms of finite Hopf algebras of type A_n* . Proceedings of the XVIth Latin American Algebra Colloquium (Spanish), 201–226, Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, (2007).
- [AS2] ———. *On the classification of finite-dimensional pointed Hopf algebras*. Ann. Math. **171**, 375–417, (2010).
- [A1] I. ANGIONO. *On Nichols algebras of diagonal type*. J. Reine Angew. Math. **683** (2013), 189–251.
- [A2] ———. *Distinguished Pre-Nichols algebras*, Transf. Groups **21**, 1–33, (2016).
- [AG] I. ANGIONO, A. GARCÍA IGLESIAS. *Pointed Hopf algebras with standard braiding are generated in degree one*, Contemp. Math. **537** (2011).
- [B] G. BERGMAN. *The diamond lemma for ring theory*. Adv. Math. **29** (1978), 178–218.
- [CG] A. M. COHEN, D. A. H. GIJSBERS. *GBNP 0.9.5 (Non-commutative Gröbner bases)*, <http://www.win.tue.nl/~amc>.
- [GAP] THE GAP GROUP, *GAP — Groups, Algorithms and Programming*. Version **4.4.12**, (2008), <http://www.gap-system.org>.
- [Gu] GÜNTHER, R., *Crossed products for pointed Hopf algebras*. Comm. Algebra, **27**, 4389–4410, (1999).
- [H] I. HECKENBERGER. *Classification of arithmetic root systems*. Adv. Math. **220** (2009), 59–124.
- [Mo] S. MONTGOMERY. *Hopf algebras and their action on rings*. CBMS Lecture Notes 82, American Math Society, Providence, RI, (1993).
- [S] SCHAUENBURG, P., *Hopf bi-Galois extensions*. Comm. Alg. **24**, 3797–3825, (1996).

FAMAF-CIEM (CONICET), UNIVERSIDAD NACIONAL DE CÓRDOBA, MEDINA ALLENDE S/N, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, REPÚBLICA ARGENTINA.
E-mail address: angiono@famaf.unc.edu.ar, aigarcia@famaf.unc.edu.ar